Decentralized Estimation of Laplacian Eigenvalues in Multi-Agent Systems

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Abstract

In this paper we present a decentralized algorithm to estimate the eigenvalues of the Laplacian matrix that encodes the network topology of a multi-agent system. We consider network topologies modeled by undirected graphs. The basic idea is to provide a local interaction rule among agents so that their state trajectory is a linear combination of sinusoids oscillating only at frequencies function of the eigenvalues of the Laplacian matrix. In this way, the problem of decentralized estimation of the eigenvalues is mapped into a standard signal processing problem in which the unknowns are the finite number of frequencies at which the signal oscillates.

Key words: Laplacian spectrum, decentralized estimation, multi-agent systems.

1 Introduction

Nowadays the research community is investing more and more effort in designing coordination and estimation algorithms for networked multi-agent systems [2,5,9,17]. The network topology of a multi-agent system can be effectively described by means of a graph, where nodes represent agents and edges represent couplings between them [15]. The emergent behavior of such a system depends both on the interactions between agents and on the network topology.

Algebraic graph theory [8] provides powerful tools to analyze a graph. As an example, the knowledge of the spectrum of the Laplacian matrix associated to a graph can be used to estimate topological properties of an undirected graph, e.g., algebraic connectivity, average degree, diameter, spectral gap, connectivity measures [14]. In the context of multi-agent systems, this knowledge may also provide powerful insights into the dynamics of a networked control system. As an example, as stated in [17], the algebraic connectivity, i.e., the second smallest eigenvalue, is a fundamental parameter to estimate the worst case convergence rate of consensus algorithms and, more in general, of multi-agent systems with local interactions described by the Laplacian matrix such as leader-follower networks [18].

Unfortunately, the spectrum of the Laplacian matrix is not readily computable in a distributed setting where the network topology is unknown. In order to overcome this limitation, we have designed a local interaction rule so that the resulting dynamical system oscillates only at frequencies corresponding to the eigenvalues of the Laplacian matrix that encodes the network topology. In this way, the problem of estimating the eigenvalues is mapped into a signal processing problem solvable independently by each agent in a decentralized fashion, using tools from signal processing or system identification theory.

Compared to the state of the art, discussed in detail in next section, our approach allows to estimate the full spectrum of the symmetric Laplacian matrix without the need to estimate all the corresponding eigenvectors. Moreover our approach provides an approximate estimation of the eigenvalues in finite time.

⋆ The research leading to these results has received funding from the European Union Seventh Framework Programme [FP7/2007-2013] under grant agreement n 257462 HYCON2 Network of excellence, by the Italian grant FIRB Futuro in Ricerca, project NECTAR, code RBFR08QWUV and by Region Sardinia, LR 7/2007 (call 2010) under project SIAR (CRP-24709).

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The contributions of this paper are the following:

- We propose a novel local interaction rule to make the network state oscillate at frequencies corresponding to the eigenvalues of the Laplacian matrix, thus mapping the decentralized eigenvalue estimation problem into a standard signal processing problem. Thanks to persistent oscillations that carry the required information, standard system identification techniques can be adopted bypassing identification issues raised by large-scale systems due to their high system order.
- We extend [6] by characterizing analytically the amplitude and phase of the oscillations as function of the eigenvectors of the Laplacian and the initial conditions.
- We propose an improvement with respect to [6] so that no component at null frequency exists in the evolution of the agents' state. The removal of the DC component allows the straightforward application of frequency estimation algorithms such as the one in [4].

Related works

In [22] Zavlanos *et al.* investigated the problem of how to coordinate a network of mobile robots with position-dependent topology so that the corresponding adjacency matrix has a given set of eigenvalues. This approach is based on artificial potentials, function of the inter-agent distances, that allow a gradient descent algorithm to make the network converge to a topology whose eigenvalues are the desired ones. In this preliminary paper the authors consider the spectral moments related to the spectrum of the adjacency matrix to be known.

In [7] Franceschelli *et al.* presented a necessary and sufficient condition to verify observability and controllability of a leader-follower network of mobile vehicles with unknown topology based on the algorithm in [6] and its extension in this paper.

In [21] Yang *et al.* proposed a technique for the estimation of the second smallest eigenvalue of a weighted Laplacian matrix based on the power iteration algorithm by the estimation of the corresponding eigenvector. In addition, the authors discuss a decentralized control algorithm to maximize the algebraic connectivity. The idea is to let agents move so that links are added or weights changed as two agents come closer.

In [19] Sahai *et al.* presented an approach building on the idea of [6] for the application of clustering. The authors propose a local interaction rule formally equivalent to the wave equation discretized in time and space and show that the "wave equation method", can be used to cluster a graph by estimating the sign of the coefficients of the discrete Fourier transform corresponding to the second smallest eigenvalue. Furthermore, their approach is superior with respect to the convergence speed to the state of the art of eigenvector based clustering algorithms.

Finally, in [10] Kempe *et al.* are interested in computing in a decentralized way an approximation of the first k eigenvectors of a symmetric matrix that encodes the network topology. Their algorithm takes inspiration from the orthogonal iteration algorithm and assumes that the network topology is unknown to the nodes. This algorithm can be adapted to our objective, i.e., the distributed estimation of the eigenvalues of the Laplacian matrix, by introducing a distributed technique for the computation of the Rayleigh quotient.

2 Online Spectrum Estimation

Let us consider the interactions of a network of agents described by an undirected graph $\mathcal{G} = \{V, \mathcal{E}\}$, where $V = \{1, \ldots, n\}$ is the set of agents and $\mathcal{E} \subseteq V \times V$ is the set of edges: an edge $e_{i,j}$ exists between agents i and j if agent i interacts with agent j.

Let $\mathcal{N}_i$ define the neighborhood of agent i, namely the set of indices of the agents connected by an edge with agent i. In particular, $|\mathcal{N}_i| = \Delta_i$ where Δ_i is called degree of agent i. Let $\mathcal{L}$ be the Laplacian matrix of graph $\mathcal{G}$, it is a $n \times n$ matrix the elements of which are $l_{ij} = \Delta_i$ if $i = j$, $l_{ij} = -1$ if $j \in \mathcal{N}_i$ and 0 otherwise. The Laplacian matrix $\mathcal{L}$ of an undirected graph is symmetric by construction and thus all its eigenvalues are real. Furthermore, for a connected graph, it has one null structural eigenvalue with corresponding eigenvector equal to the vector of ones $\mathbf{1}_n$ of appropriate dimensions, thus $\mathcal{L}\,\mathbf{1}_n = \mathbf{0}_n$. In addition, according to the Gershgorin disc theorem, a symmetric Laplacian has all its eigenvalues located within $[0, 2\,\Delta_{max}]$, where Δ_{max} is the maximum degree between the agents in the graph.

We now present a decentralized algorithm to estimate the eigenvalues of the Laplacian matrix. The algorithm requires each agent i to store two variables $x_i, z_i \in \mathbb{R}$ and apply a local update rule upon receiving the values of the equivalent variables from its neighbors.

Algorithm 1 (Online Spectrum Estimation)

(1) *Each agent sets $t = 0$ and chooses an initial condition uniformly at random $x_i(0), z_i(0) \in \{-1, 1\}$.*

(2) *Each agent* simulates *the following local interaction rule with its neighbors $\mathcal{N}_i(t)$*

$$\begin{cases} \dot{x}_i(t) = z_i(t) + \sum_{j\in\mathcal{N}_i} \left(z_i(t) - z_j(t)\right), \\ \dot{z}_i(t) = -x_i(t) - \sum_{j\in\mathcal{N}_i} \left(x_i(t) - x_j(t)\right). \end{cases} \tag{1}$$

(3) *In a time window of length T, agent i estimates the frequencies of the sinusoids of which signal $x_i(t)$ is composed.*

(4) *The values of the frequencies estimated correspond to the eigenvalues of the Laplacian matrix $\mathcal{L}$ shifted by 1 and are given as output.* ■

Note that *Step* 3 can be solved by several methods of signal processing or system identification. In particular, the required value of T depends on the chosen algorithm. In this paper, as discussed in Section 3, we exploit the method presented in [4] to implement Algorithm 1.

The behavior of the network when all the agents update their state according to eq. (1) can be described as follows

$$\begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} = \mathcal{A} \cdot \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} \mathbf{0}_{n\times n} & I+\mathcal{L} \\ -I-\mathcal{L} & \mathbf{0}_{n\times n} \end{bmatrix} \tag{2}$$

where I is the $n \times n$ identity matrix and $\mathbf{0}_{n\times n}$ is the null $n \times n$ matrix. Note that for any network topology $\mathcal{A}$ is skew symmetric, i.e., $\mathcal{A}^T = -\mathcal{A}$. In the following theorem, we prove that the eigenvalues of $\mathcal{A}$ can be analytically derived from the eigenvalues of the Laplacian matrix $\mathcal{L}$ and they are all structurally purely imaginary.

Lemma 1 *Let $\mathcal{G}$ be an undirected graph with Laplacian $\mathcal{L}$. Let matrix $\mathcal{A}$ be defined as in eq.* (2)*. To any eigenvalue $\lambda_{\mathcal{L}}$ of $\mathcal{L}$ it corresponds a couple of complex and conjugate eigenvalues $\lambda_{\mathcal{A}}, \bar{\lambda}_{\mathcal{A}}$ of $\mathcal{A}$, that is:*

$$\lambda_{\mathcal{A}} = j(1+\lambda_{\mathcal{L}}), \quad \bar{\lambda}_{\mathcal{A}} = -j(1+\lambda_{\mathcal{L}}),$$

while the corresponding eigenvectors $v_{\lambda_{\mathcal{A}}}$ are function of the eigenvectors $v_{\lambda_{\mathcal{L}}}$ of $\mathcal{L}$

$$v_{\lambda_{\mathcal{A}}} = \begin{bmatrix} v^T_{\lambda_{\mathcal{L}}} & jv^T_{\lambda_{\mathcal{L}}} \end{bmatrix}^T, \qquad \bar{v}_{\bar{\lambda}_{\mathcal{A}}} = \begin{bmatrix} v^T_{\lambda_{\mathcal{L}}} & -jv^T_{\lambda_{\mathcal{L}}} \end{bmatrix}^T.$$

Proof: See Appendix A. □

By Lemma 1 it follows that the state of each agent has an oscillatory trajectory. Furthermore, as detailed by Theorem 2, this trajectory is a linear combination of sinusoids oscillating only at frequencies function of the eigenvalues of the matrix Laplacian. In the following we assume the Laplacian to have m *distinct* eigenvalues labeled as follows: $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_m$.

Theorem 2 *Let us consider a system described by eq.* (2) *relative to a network whose graph $\mathcal{G}$ is connected. Let $x(0) = x_0$ and $z(0) = z_0$ be the state initial conditions. Let $\delta(\cdot)$ be the Dirac's delta function. Let λ_j be an eigenvalue of the Laplacian matrix $\mathcal{L}$ of graph $\mathcal{G}$ with algebraic multiplicity ν_j and let m be the number of distinct eigenvalues. Let v_1 be the unitary norm eigenvector corresponding to $\lambda_1 = 0$, and $v_j^{(k)}$, $k = 1,\dots,\nu_j$, be the ν_j unitary norm eigenvectors associated to $\lambda_j > 0$. The module of the Fourier transform of the i-th state components $x_i(t)$ and $z_i(t)$, $i = 1,\dots,n$, can be written as*

$$|\mathcal{F}[x_i(t)]| = |X_i(f)| = \sum_{j=1}^{m} \frac{a_{j,i}}{2}\,\delta\left(f \pm \frac{1+\lambda_j}{2\pi}\right),$$

$$|\mathcal{F}[z_i(t)]| = |Z_i(f)| = \sum_{j=1}^{m} \frac{b_{j,i}}{2}\,\delta\left(f \pm \frac{1+\lambda_j}{2\pi}\right),$$

where f is the frequency domain variable. In addition, the coefficients $a_{j,i}$ and $b_{j,i}$ are given by

– *For $\lambda_1 = 0$*

$$\begin{aligned} a_{1,i} &= v_1(i)\, v_1^T x(0) = \frac{\mathbf{1}_n^T\, x(0)}{n}, \\ b_{1,i} &= v_1(i)\, v_1^T z(0) = \frac{\mathbf{1}_n^T\, z(0)}{n}. \end{aligned} \tag{3}$$

– *For $\lambda_j > 0$*

$$a_{j,i} = b_{j,i} = \sqrt{\begin{array}{c} \left[\sum_{k=1}^{\nu_j}\left(v_j^{(k)}(i){v_j^{(k)}}^T x(0)\right)\right]^2 + \\ \left[\sum_{k=1}^{\nu_j}\left(v_j^{(k)}(i){v_j^{(k)}}^T z(0)\right)\right]^2 \end{array}}. \tag{4}$$

Proof: See Appendix B. □

The above theorem states the key result of this paper. In fact, it implies that each agent can independently solve the problem estimating the eigenvalues by estimating the frequencies at which its own state variable $x_i(t)$ oscillates.

Remark 1 *Few important remarks are now in order:*

- *The value of $x_i(t)$ can be seen as the output of the i-th agent. If the system is not observable from the output $x_i(t)$ then some coefficients a_j are null and thus the corresponding mode cannot be detected by agent i.*
- *For each agent, the amplitude of the sinusoid oscillating at $\omega = \lambda_1 = 1$ corresponds to the instantaneous average of the state variables.* ■

The observability and controllability of a system, the dynamics of which are described by the Laplacian matrix, have been studied in [12,13,1] from a graph theoretical point of view. In the following theorem we show that the observability property of the Laplacian matrix is equivalent to that of system (2) given an appropriate output matrix. The proposed method is capable of estimating all the eigenvalues corresponding to modes of system (2) that are observable from the state of a given agent, therefore the following theorem characterizes when the method can estimate all the eigenvalues.

Theorem 3 *Let $\mathcal{A}$ be the matrix describing the group dynamics as in* (2)*. Let C be a $k \times n$ output matrix, with $k \in \mathbb{N}$. Let*

$$\mathcal{A} = \begin{bmatrix} \mathbf{0}_{n\times n} & I+\mathcal{L} \\ -I-\mathcal{L} & \mathbf{0}_{n\times n} \end{bmatrix} \quad \text{and} \quad \hat{C} = \begin{bmatrix} C & \mathbf{0}_{k\times n} \\ \mathbf{0}_{k\times n} & C \end{bmatrix}.$$

Let $\mathcal{O}_{\mathcal{A}} = \mathcal{O}(\mathcal{A}, \hat{C})$ and $\mathcal{O}_{\mathcal{L}} = \mathcal{O}(\mathcal{L}, C)$ be the observability matrices built with the corresponding matrices. Then:

$$Rank\left(\mathcal{O}_{\mathcal{A}}\right) = 2Rank\left(\mathcal{O}_{\mathcal{L}}\right).$$

Proof: See Appendix C. □

Remark 2 *We point out that information about the observability of the system is not required for the execution of Algorithm 1. In [7] we proved that for a network of n agents, the estimation of n distinct eigenvalues by a single agent is a sufficient and necessary condition for observability and by duality controllability of the network by the same agent. Hence, if the number of agents n is known, each agent can verify the observability property of the network by itself.* ■

We now state the main result of this paper that proves the correctness of Algorithm 1.

Theorem 4 *Consider a connected network $\mathcal{G}$ of n agents that executes Algorithm 1. Let the initial conditions of system* (2) *be not orthogonal to any eigenvector of matrix $\mathcal{L}$. Let $C = [0\ldots,1,\ldots,0]$ be zero everywhere except for the i-th unitary element, with $i \in V$. If $\mathcal{O}\left(\mathcal{L}, C\right)$ is full rank, then agent i can estimate all the eigenvalues of the Laplacian matrix.*

Proof: Due to Lemma 1 all the eigenvalues of system (2) are purely imaginary and correspond to the eigenvalues of the Laplacian matrix shifted by one. Furthermore, if the initial conditions are not orthogonal to all the eigenvectors of $\mathcal{L}$ and $\mathcal{O}\left(\mathcal{L}, C\right)$ is full rank as discussed in Theorem 3, then all the sinusoids corresponding to the system modes have coefficients strictly greater than zero. Thus by applying a frequency estimation algorithm to the signal $x_i(t)$, for instance the one in [4], agent i can estimate the full spectrum of the Laplacian matrix by only observing its own state evolution. □

It is relevant to point out that even if the system is not observable from a single agent perspective, it will always be observable if matrix C is the identity matrix, i.e., if we consider all the information that agents locally retrieve.

3 Numerical implementation of the approach

The system in eq. (2) is a marginally stable linear system since all its eigenvalues lie on the imaginary axis. The stability of a system with eigenvalues exactly on the imaginary axis is not considered to be robust because even the slightest parameter uncertainties may render the system unstable. In our case there is no parameter uncertainty because system (2) is based on the Laplacian matrix the elements of which depend only on the existence of links between the agents. Thus, for any network topology system (2) can not be stable or unstable

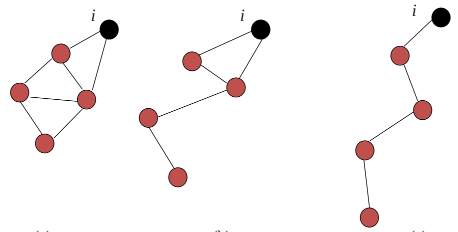

Fig. 1. Topology variation with respect to time for a network composed of 5 agents.

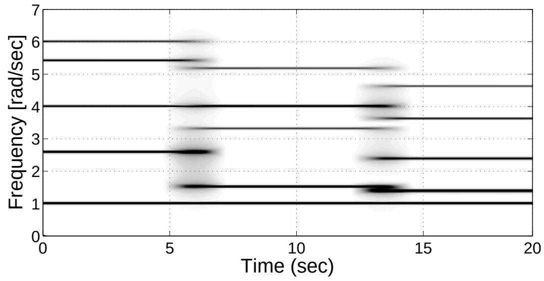

Fig. 2. Spectrogram of the time varying topology shown in Fig. 1 computed by the ith agent with respect to the output associated to its state variable $x_i(t)$.

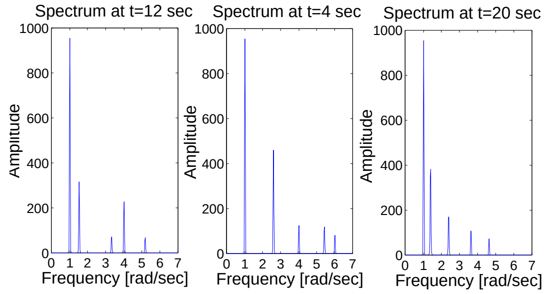

Fig. 3. Spectrum of the time varying topology shown in Fig. 1 computed by the ith agent at time $t = 4$, $t = 12$, $t = 20$.

but only marginally stable. Furthermore we point out that since no sensing/measurement is involved, no noise is generated from the application of the local interaction rule.

In this paper, we implemented the approach in [4] to estimate the frequencies at which the signal oscillates. Furthermore we performed a spectral analysis by means of the Discrete Fourier Transform (DFT).

3.1 Approximate Frequencies Estimation Method

The problem of estimating the frequencies of a signal such as

$$y(t) = \sum_{i=1}^{n} A_i sin(\omega_i t + \phi_i) \tag{5}$$

has been extensively studied in control theory via offline methods based on Fourier analysis tools and online methods [11,16]. In this paper, the approximate

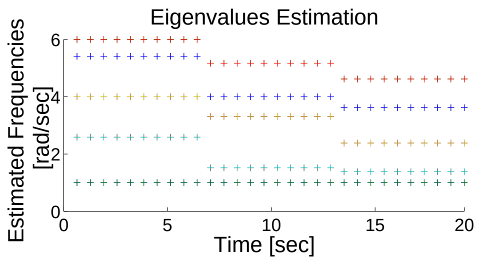

Fig. 4. Eigenvalues estimated every $T = 1$ units of time by the approximate frequency estimation method in Subsection 3.1. Note that the frequencies are shifted by 1 with respect to the eigenvalues of $\mathcal{L}$ in Table 1.

	$t = 5$		$t = 15$		$t = 25$	
	$\lambda_{\mathcal{L}}$	$\hat{\lambda}_{\mathcal{L}}$	$\lambda_{\mathcal{L}}$	$\hat{\lambda}_{\mathcal{L}}$	$\lambda_{\mathcal{L}}$	$\hat{\lambda}_{\mathcal{L}}$
λ_1	0	0	0	0	0	0
λ_2	1.5857	1.5857	0.5188	0.5188	0.3819	0.3819
λ_3	3.0000	2.9998	2.3111	2.3110	1.3819	1.3819
λ_4	4.4142	4.4138	3.0000	2.9998	2.6180	2.6179
λ_5	5.0000	4.9987	4.1700	5.1696	3.6180	3.6177

Table 1
Comparison between the actual spectrum of the Laplacian, denoted by $\lambda_{\mathcal{L}}$, and the estimated spectrum, denoted by $\hat{\lambda}_{\mathcal{L}}$ of the time varying topology shown in Fig. 1.

frequency estimation algorithm in [4] has been implemented. It allows to estimate the frequencies of a signal of the form in eq.(5) by assuming an upper bound `nMax` to the number of existing frequencies to be available. The input of the algorithm is the sampling time T_s, an upper bound `nMax` of the number of expected frequencies, and a measure of the approximation error S_e. Furthermore, the length of the time window considered must be greater than the largest period of the sinusoid with the smallest frequency.

The output of the algorithm is the number `n` of estimated frequencies and their values $\{1 + \lambda_{\mathcal{L}}\}$ and a flag `f` the value of which is `true` if the percentage error when reconstructing the signal with the estimated coefficients fits the threshold S_e, `false` otherwise. Note that, a great advantage of this algorithm is that the estimation can be worked out in finite time. If an observer with asymptotic convergence is required, the output of the algorithm described above could be used as input for the algorithm proposed in [3].

3.2 Simulations with switching topology

In order to corroborate the mathematical results, simulations have been carried out by exploiting the 4th Order Runge-Kutta Method (RK4) to simulate the system (2). Regarding the signal processing, let us recall that this can be always carried out locally by each agent in spite of the particular technique adopted.

In the simulation, a network of agents the topology of which changes over time is considered. In detail, Fig. 1-a) depicts the network topology at the time interval $t \in [0, 6.4)$, Fig. 1-b) describes the network topology at the time interval $t \in [6.4, 12.9)$ and Fig. 1-c) describes the network topology at the time interval $t \in [12.9, 20]$. Each agent is running the interaction rule described in eq. (1).

Fig. 2 shows the spectrogram of the time varying topology computed by the agent i with respect to its state variable $x_i(t)$. The spectrogram was computed by this agent with $f_s = \frac{100}{2\pi}$. The x and y axes of the spectrogram represent respectively the time step and angular frequency, while the color of the spectral line describes the amplitude of the frequency peaks, i.e., white means zero amplitude while black means an amplitude greater than 0.1.

Fig. 3 shows a section of the spectrogram at different time steps, namely $t = \{4, 12, 20\}$, representing the spectrum of the three network topologies taken into account.

To this example, we also applied the method in Section 3.1 to estimate the frequency of the sinusoids and thus the eigenvalues of the Laplacian. The comparison between the eigenvalues of the Laplacian matrix of the time-varying network topology in Fig. 1 and the estimated eigenvalues in Fig. 4 using the approximate frequency estimation method in Subsection 3.1 is shown in Table 1. This method was implemented choosing as sampling frequency $f_s = \frac{100}{2\pi}$ that is twenty times the maximum expected frequency in the signal. This frequency corresponds to the largest eigenvalue of the Laplacian matrix plus one. The length of the time window used to computed each estimation is $T = 1$ sec, which is the period of the slowest sinusoid.

3.3 Computational Cost

To study the computational cost of Algorithm 1 we adopt the metrics proposed in [10], i.e., we count the number of communication rounds required to obtain an estimation with a certain accuracy. In this view, two important aspects must be considered: (i) the proposed algorithm consists in a local interaction rule that is supposed to be applied continuously; (ii) the required signal processing is carried out locally by an agent and several techniques can be adopted. Therefore, the computational cost analysis consists in the study of the cost of simulating the local interaction rule and the cost of the signal processing. Since the study of the computational cost for the signal processing required to estimate a discrete number of frequencies contained in a signal is not the scope of this paper we focus our attention to the number of communication rounds required for the discrete time simulation of system (2) by the agents to collect enough data for the consecutive signal processing. In particular, the simulation time needed to collect a sufficient amount of data must be greater than the largest period of the sinusoid with the smallest frequency, $T_{\min}$. By considering an accurate numerical simulation method such as the fourth order Runge-Kutta method, 4 messages have to be exchanged between each agent and any of its neighbors to compute a sample of the state trajectory. It follows that for each agent the rounds of communication required to collect a sufficient amount of data can be bounded by $4 \cdot \Delta_{\max} \cdot T_{\min} \cdot f_s$, with $\Delta_{\max}$ the maximum degree in the network and f_s the chosen sampling frequency that has to be at least greater than twice the largest eigenvalue of the Laplacian, thus greater than $2\Delta_{max}$, to avoid aliasing issues.

4 Conclusions

In this paper a decentralized algorithm to estimate the Laplacian spectrum of an undirected graph has been proposed. Each agent interacts with its neighbors so that its state oscillates at the frequencies corresponding to the eigenvalues of the Laplacian matrix that encodes the network topology. Therefore, the problem of estimating the eigenvalues has been reduced to a simple and widely studied problem of signal processing which involves the estimation of the discrete number of frequencies at which the generated signal is oscillating. A theoretical analysis of the proposed technique along with numerical simulations has been provided.

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A Proof of Lemma 1

By definition, the eigenvalues of $\mathcal{A}$ are the solutions of

$$det(\mathcal{A} - \lambda I) = det\left(\begin{bmatrix} -\lambda I & I + \mathcal{L} \\ -I - \mathcal{L} & -\lambda I \end{bmatrix}\right) = 0.$$

Since $\mathcal{A}$ is a block matrix whose blocks commute [20], then

$$det(\mathcal{A}-\lambda I) = det\left(\lambda^2 I - (I+\mathcal{L})^2\right). \tag{A.1}$$

Now, denote $\lambda_{\mathcal{L}}$ the generic eigenvalue of the matrix Laplacian, it is $det(\mathcal{L}-\lambda_{\mathcal{L}} I) = 0$. By adding and subtracting the identity matrix and exploiting the fact that the eigenvalues of the square of a matrix are squared, we obtain $det((I+\mathcal{L})^2 - (1+\lambda_{\mathcal{L}})^2 I) = 0$. Thus, by (A.1), it is $\lambda^2 = -(1+\lambda_{\mathcal{L}})^2$ and $\lambda = \pm j(1+\lambda_{\mathcal{L}})$. Therefore, to each eigenvalue of the Laplacian matrix $\mathcal{L}$ it corresponds two imaginary and conjugate eigenvalues of matrix $\mathcal{A}$ denoted by $\lambda_{\mathcal{A}}$ and $\bar{\lambda}_{\mathcal{A}}$, so that

$$\lambda_{\mathcal{A}} = j(1+\lambda_{\mathcal{L}}), \quad \text{and} \quad \bar{\lambda}_{\mathcal{A}} = -j(1+\lambda_{\mathcal{L}}),$$

thus proving the first statement. Now, by definition, the eigenvectors of $\mathcal{A}$ relative to $\lambda_{\mathcal{A}}$ are solutions of

$$\begin{bmatrix} \mathbf{0}_{n\times n} & I+\mathcal{L} \\ -I-\mathcal{L} & \mathbf{0}_{n\times n} \end{bmatrix} \cdot \begin{bmatrix} v' \\ v'' \end{bmatrix} = \lambda_{\mathcal{A}} \begin{bmatrix} v' \\ v'' \end{bmatrix}$$

for which a possible solution is $v_{\mathcal{A}} = [v_{\lambda_{\mathcal{L}}}^T \quad jv_{\lambda_{\mathcal{L}}}^T]^T$. The same argument holds for the conjugate eigenvalue $\bar{\lambda}_{\mathcal{A}} = -j(1+\lambda_{\mathcal{L}})$ for which a possible solution is $\bar{v}_{\mathcal{A}} = [v_{\lambda_{\mathcal{L}}}^T \quad -jv_{\lambda_{\mathcal{L}}}^T]^T$.

B Proof of Theorem 2

When referring to the eigenvalues and eigenvectors of $\mathcal{L}$, $\lambda_{\mathcal{L},j}$ and $v_{\lambda_{\mathcal{L},j}}$ for $j = 1,\ldots,n$, we drop the subscripts $\mathcal{L}$ and $\lambda_{\mathcal{L}}$, respectively, and refer to them as λ_j and v_j for $j = 1,\ldots,n$. By Lemma 1 to each Laplacian eigenvalue λ_j it corresponds a couple of pure imaginary eigenvalues of $\mathcal{A}$ equal to $\lambda_{\mathcal{A}}, \bar{\lambda}_{\mathcal{A}} = \pm j(1+\lambda_j)$. Therefore, the state trajectory $x_i(t)$ of each agent is a linear combination of sinusoids whose amplitudes and phase shifts are function of the initial conditions and of the graph topology.

Now, we compute the coefficients of the module of the Fourier transform of $x_i(t)$. Since $\mathcal{A}$ is skew symmetric, it is a normal matrix. Due to the Spectral Theorem it is always diagonalizable through a unitary matrix [1] and all the eigenvalues have geometric multiplicity equal to their algebraic multiplicity. Thus $\mathcal{A}$ can be decomposed as $\mathcal{A} = VDV^*$, where D is diagonal with elements arranged as $D = \mathrm{diag}\{j\lambda_1, \ \ldots, \ j\lambda_n, \ -j\lambda_1, \ \ldots, \ -j\lambda_n\}$, and V is a complex matrix whose columns are the eigenvectors of $\mathcal{A}$. Furthermore, applying Lemma 1, matrix V is

$$V = \begin{bmatrix} v_1 & v_2 & \ldots & v_n & v_1 & v_2 & \ldots & v_n \\ jv_1 & jv_2 & \ldots & jv_n & -jv_1 & -jv_2 & \ldots & -jv_n \end{bmatrix}.$$

[1] A unitary matrix U is a complex matrix such that $U^*U = UU^* = I$, where U^* is the complex conjugate of U.

In the following it is assumed that v_j, $j = 1,\ldots,n$, are normalized eigenvectors such that $\|v_j\| = 1$ and $VV^* = I$. Thus

$$\left\| \alpha \begin{bmatrix} v_j^T & jv_j^T \end{bmatrix}^T \right\| = 1, \quad \alpha \in \mathbb{R}.$$

By simple manipulations we find $\alpha = \frac{1}{\sqrt{2}}$. The state trajectories of the system are captured by the matrix exponential which in our case takes the form $e^{\mathcal{A}t} = Ve^{Dt}V^*$. It follows that the state trajectory of $x_i(t)$ and $z_i(t)$ have the following form

$$\begin{aligned} x_i(t) = \sum_{j=1}^{n} &\left[v_j(i)\left(\cos((1+\lambda_j)\,t)v_j^T x(0)\right.\right. \\ &+ \left.\left.\sin((1+\lambda_j)\,t)v_j^T z(0)\right)\right], \\ z_i(t) = \sum_{j=1}^{n} &\left[v_j(i)\left(-\sin((1+\lambda_j)\,t)v_j^T x(0)\right.\right. \\ &+ \left.\left.\cos((1+\lambda_j)\,t)v_j^T z(0)\right)\right]. \end{aligned}$$

Thus, according to the notation of Theorem 2, by simple manipulations we can obtain the expression for the coefficients $a_{j,i}$ and $b_{j,i}$ associated to the eigenvalue λ_j for the i-th agent stated in eq. (3) and eq. (4), proving the theorem.

C Proof of Theorem 3

Let $\mathcal{O}_{\mathcal{A}} = \mathcal{O}(\mathcal{A},\hat{C})$, $\mathcal{O}_{I+\mathcal{L}} = \mathcal{O}(I+\mathcal{L},C)$ and $\mathcal{O}_{\mathcal{L}} = \mathcal{O}(\mathcal{L},C)$ be the observability matrices of the corresponding matrices. It can be shown by row permutation that:

$$\begin{aligned} &\mathrm{Rank}\,(\mathcal{O}_{\mathcal{A}}) = \\ &\mathrm{Rank}\left(\begin{bmatrix} \mathcal{O}_{I+\mathcal{L}} & \mathbf{0}_{n\times n} \\ \mathcal{O}_{I+\mathcal{L}}(I+\mathcal{L})^n & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathcal{O}_{I+\mathcal{L}} \\ \mathbf{0}_{n\times n} & \mathcal{O}_{I+\mathcal{L}}(I+\mathcal{L})^n \end{bmatrix}\right) \end{aligned}$$

hence it also holds that

$$\mathrm{Rank}\,(\mathcal{O}_{\mathcal{A}}) = \mathrm{Rank}\left(\begin{bmatrix} \mathcal{O}_{I+\mathcal{L}} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathcal{O}_{I+\mathcal{L}} \end{bmatrix}\right)$$

from which it follows that $Rank\,(\mathcal{O}_{\mathcal{A}}) = 2Rank\,(\mathcal{O}_{I+\mathcal{L}})$.

Finally, by noticing that the eigenvalues of the matrices $\mathcal{O}_{I+\mathcal{L}}$ and $\mathcal{O}_{\mathcal{L}}$ are related as $\lambda_{I+\mathcal{L}} = 1+\lambda_{\mathcal{L}}$ and share the same eigenvectors, from the PBH observability test it follows that

$$Rank\,(\mathcal{O}_{\mathcal{L}}) = Rank\,(\mathcal{O}_{I+\mathcal{L}}) = 2Rank\,(\mathcal{O}_{\mathcal{L}}).$$