A distributed framework for k-hop control strategies in large-scale networks based on local interactions

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***Abstract*— In this paper, we propose a distributed framework for large scale networks to attain control strategies requiring k-hop interactions. This research is motivated by the observation that in many practical applications and operational domains involving large-scale networks, such as environmental monitoring or traffic load balancing, agents may be required to collect only information concerning other agents located sufficiently close to them, that is agents topologically at most k-hop away. In this setting, distributed observers available at the state of art, which typically estimates the full network state, may be inadequate due to scalability issues. Differently, we propose a distributed finite-time observer which allows each agent to estimate the state of its k-hop neighbors by interacting only with the agents belonging to its 1-hop neighborhood. Furthermore, we demonstrate that for feedback control strategies based on k-hop neighborhood information, which are Input-to-State stable, the proposed distributed finite-time observer can be effectively used to design stable large-scale networked control strategies. Numerical results are provided to corroborate the theoretical findings.**

I. Introduction

Large-scale Networked Systems (LSNSs) represent an effective modeling for a variety of application settings ranging from power grids to communication networks [?], [?], [?], [?], [?], [?]. Research in the field of modeling, analysis and control of LSNSs has been an attractive field for the control community for a long time [?], [?], [?]. Over the years, several definitions of LSNSs have been proposed where differences are mostly related to the control organizational structures, moving from hierarchical control structures to fully decentralized control structures [?], [?], [?].

Nowadays, inspired by the technological advances of, for examples, Microelectromechanical systems (MEMS), the trend is towards the development of fully distributed locally interacting systems. More specifically, we are interested in an application setting where each node (agent) is characterized by limited sensing and communication capabilities, and interactions are limited to pairs of agents that are within the mutual range of communication. As a matter of fact, given the distributed nature of the system, the exchange of information becomes crucial to realize almost any collaborative objective, ranging from agents coordination, to data fusion and signal tracking. The underlying idea is to make up for the lack of global knowledge by means of pairwise interactions in such a way that the global information can be either retrieved or compensated by means of local information exchange. This working paradigm has driven the research activities of the last two decades in the distributed control community [?], [?], [?], [?].

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This work is motivated by the observation that in many practical applications and operational domains involving large-scale networks, such as environmental monitoring or traffic load balancing, due to the locality of the phenomena, the adoption of a suitable local strategy by each agent may require to collect only information concerning other agents located sufficiently close to them, that is agents topologically at most k-hop away. In other words, control strategies to attain collaborative behavior may reasonably be developed by having each agent relying only on a partial knowledge of the overall state of the network. In such a context, typical approaches to estimate the state of a system, such as distributed observers, may perform poorly as they were originally designed to estimate the state of the whole network [?], while each agent may only be interested in retrieving a portion of such state related to the agents that are at most k-hop away from it. For this reason, we developed a distributed finite-time observer which allows each agent to estimate the state of its k-hop neighbors by interacting only with the agents belonging to its 1-hop neighborhood. Compared to distributed state observers available at the state of the art, this significantly reduces the computational burden when applied to large-scale networks, that is networks where the number of agents is considerably large.

Furthermore, we demonstrate that for feedback control strategies requiring k-hop neighborhood information, which are Input-to-State stable, the finite-time convergence property of the proposed distributed observer allows to safely close the control loop around the estimated k-hop state information. Indeed, this allows to decouple the design of the control law from the stability analysis of the multi-agent dynamics when closing the loop through the state observer. Thus demonstrating that the proposed distributed finite-time observer can be effectively used to attain stable collaborative large-scale networked systems.

To summarize, in this paper the following contributions are made:

- we provide a very general framework to design collaborative control strategies on large scale systems by abstracting away from the locality of the interactions. To this aim, we propose a distributed finite-time observer which allows each agent to estimate the state of its k-hop neighbors by interacting only with the agents belonging to its 1-hop neighborhood;
- we demonstrate that if the closed-loop dynamics of the multi-agent system with a given feedback controller is Input-to-State Stable (ISS) under the assumption of global knowledge of the k-hop neighbors state, then the system remains stable also while using the devised observer in the case such k-hop knowledge is not available; thus highlighting a sort of decoupling principle between control design and observer design.

The rest of the paper is organized as follows. In Section II related works are discussed. In Section III some required mathematical background is provided, while Section IV describes the problem setting. The proposed k-hop Graph-Based State Observer is described in Section V, and Section VI carries out a closed-loop analysis in the case the estimated state is used within a feedback control strategy. In Section VII simulations results are described. Finally, in Section VIII conclusions are drawn and future work is discussed.

II. Related Work

The problem of designing distributed observers for large-scale networks of interconnected systems represents a hot research topic with solutions differing, for example, in terms of agent dynamics, communication constraints, size of the network and convergence properties of the estimated information. Such problem has been studied since the 70s [?], [?], [?]. In particular, in [?] the authors suggest to decompose a large linear system into a number of interconnected subsystems with decentralized (scalar) inputs or outputs. The underlying idea is that, by means of such a decomposition, classical techniques can be used to either stabilize or observe the subsystems. The problem of designing a decentralized observation scheme with a specified convergence rate is addressed in [?]. In this work, the large-scale system is described as an interconnection of several reduced-order subsystems. Work [?] proposes an algorithm for the design of decentralized observation schemes in large-scale interconnected systems which is based on the notion of block diagonal dominance in matrices describing the system dynamics. In particular, the authors demonstrate how the observer gains can be tailored systematically to the existing interconnection pattern within the overall system.

In addition to these seminal works, several recent results have appeared concerning distributed state-estimation and control for general networked systems based on inter-agent communication and consensus theory [?], [?], [?], [?], [?].
In this perspective, network decomposition in sub-systems, each one with its own local controller, is introduced in [?]. In this work, information exchange may or may not be allowed depending on the network constraints. In [?] a consensus-based decentralized observer for a team of agents described by a discrete-time linear dynamics is proposed. Locally available information and the knowledge of the plant model are exploited to estimate the overall plant state. Similarly, in [?] the authors propose an asymptotic observers where each agent exploits local state measurements and the state communicated by its 1-hop neighbors with the objective of estimating the global state of the network. Finally, the problem of distributed output regulation for heterogeneous nonlinear networked agents is addressed in [?], where local measurement of the agent output and the exchange of some local information among different observers are used to asymptotically estimate some state variables. Notably, while these approaches were originally developed for networked systems where the per-agent controller would depend on the full network state, their applicability to the context of large-scale networks may be debatable, due to their scalability issue with respect to the network dimensionality and the fact that a full knowledge of the network state may even not be required.

To the best of our knowledge, solutions specifically devised for large-scale systems are sparse. Recent results include [?] and [?] where distributed Kalman filtering techniques are used, and [?] where both linear and nonlinear interconnected systems are considered.

Focusing on recent results concerning large scale-network, work in [?] faces the problem of stabilizing networked control systems with sparse observer-controller networks. In particular, first the authors derive a set of stability conditions based on the Lyapunov direct methods, then they exploit these conditions to derive a low-complexity algorithm for the design of a sparse observer for linear time-invariant (LTI) networked control systems with arbitrary topology.
A distributed observer to estimate the states of a large scale network of semi-linear systems interconnected by a positive time varying coupling strength is presented in [?]. Briefly, the authors design a network of *local* observers which requires only 1-hop local node level information and exchanges their local state estimates with their *neighboring* observers. The underlying idea of this approach is to minimize the number of measurements from the network to reduce the sensor requirements. In [?], the authors investigate the design of a distributed guaranteed-cost controller based on information derived from a local distributed observer for linear systems with nonlinear interconnections. First the dynamics of the estimation error is written as a class of linear parameter varying (LPV) systems by resorting to the differential mean value theorem; then, necessary and sufficient conditions for the existence of distributed observer-based guaranteed cost controllers are synthesized based on a linear matrix inequality (LMI) approach.

One important performance index of distributed protocols is the convergence rate. Most of the aforementioned works guarantee asymptotic convergence which means that the estimated information approaches the true value only as time approaches to infinity. However, finite-time convergence is a desirable property when fast convergence and precision is required. In this regard, finite-time observers have been proposed in recent years. For example, the work in [?] proposed a solution to the finite-time consensus problem. The adopted protocol is based on a distributed observer with finite settling theoretically estimated. In [?], observer-based distributed controllers are proposed for both directed and undirected networks with the aim of achieving coordinated tracking. A finite-time distributed observer is designed to the scope and it is shown to work when the relative state or relative output of neighbour agents is available. A robust finite-time discontinuous observer is designed in [?] with the purpose of solving a multi-agent leader-follower problem. In [?], the authors propose a nonlinear distributed observer with finite-time convergence property for linear time-invariant (LTI) network dynamics in large-scale system.

An observer-based control scheme for networked system is proposed in [?]. The present work significantly differs from [?] where agents were described by a simple linear first order dynamics, the cooperative task was described by a linear-in-the-state function which depends on the whole state of the network, the proof of asymptotic convergence of the controller-observer schema was carried out by considering all together the agent dynamics, the observer structure and the very specific control law and, finally, the approach was not meant to scale with the number of agents.
Moreover, a preliminary version of this work can be found in [?] from which the present work differs for the more rigorous analysis, the more general system model considered, the ISS-based closed loop analysis.

III. Preliminaries

A. Network Modeling

Let us consider a multi-agent system composed of n interacting agents. Let the information exchange among the agents be described by an undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} = \{1, \ldots, n\}$ is the set of agents and $\mathcal{E} = \{(i, j)\}$ is the set of pairwise interactions among agents i and j.

Let us define a path between agents i and j as the set of edges through which an agent j can be reached by an agent i; in the following we will denote as k-hop path between agents i and j, a path which involves k edges from agent i to reach agent j. Note that since the graph is undirected, the indexed i and j can be arbitrarily switched. Let $\mathcal{N}_i^k$ denote the k-hop neighborhood of an agent i, that is the set of agents j for which there exists a p-hop path from agent j to i with $p \leq k$. Let us now introduce the augmented k-hop neighborhood $\bar{\mathcal{N}}_i^k$ as the k-hop neighborhood of agent i including the agent i itself, that is $\bar{\mathcal{N}}_i^k = \mathcal{N}_i^k \cup \{i\}$, with cardinality η_i ($\eta_i = |\bar{\mathcal{N}}_i^k|$). Moreover, let as denote the elements of this set as $\mathcal{V}_i = \bar{\mathcal{N}}_i^k = \{n_i^1, \ldots, n_i^{\eta_i}\}$, where n_i^j is the global index of the j-th neighbour of agent i . Let us define the adjacency matrix $W \in \mathbb{R}^{n \times n}$ as the matrix where

$$W = \{w_{ij}\} : \quad w_{ii} = 0, \quad w_{ij} = \begin{cases} 1 & \text{if } (j, i) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}$$

for which an element w_{ij} is different from zero if node j can send information to node i. The degree matrix $D = \operatorname{diag}\{d_1, \ldots, d_n\}$ is defined as the diagonal matrix where $d_i = |\mathcal{N}_i^1|$; while the Laplacian matrix associated to the graph is defined as $L = D - W$. As far as the Laplacian matrix is concerned, zero is always an eigenvalue with $1_n \in \mathbb{R}^n$ as the corresponding right eigenvector, i.e., $L1_n = 0_n$, with 1_n (0_n) the n-dimensional column vector with all elements equal to 1 (0). Hence, $\operatorname{rank}(L) \leq n - 1$ where the equality holds when the graph is connected [**?**].

Finally, let us denote with $\otimes$ the Kronecker product, and with $Q \succ 0$ ($Q \prec 0$) a positive (negative) definite matrix. In addition, given a positive definite matrix Q, the symbol $\sigma(Q)$ denotes the spectrum of its matrix argument, while $\lambda_j(Q)$, $\lambda_{min}(Q)$ and $\lambda_{max}(Q)$ are the generic j th, the minimum and maximum eigenvalues of Q.

B. Nonsmooth Analysis

We now briefly review the Filipov solution concept for differential equations with discontinuous right-hand side, the nonsmooth analysis of Clarke's Generalized Gradient, and the chain-rule for differentiating regular functions along Filipov solution trajectories. The reader is referred to [**?**], [**?**], [**?**] and references therein for a comprehensive overview of the topic.

Consider the following differential equation

$$\dot{x} = f(x) \tag{1}$$

with $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ a measurable and essentially locally bounded function. First, we need to clarify what it means to be a solution of this equation.

Definition 1 (Filipov Solution)**.** A vector function $x(\cdot)$ is called solution of (1) on a time interval $[t_0, t_i]$ if $x(\cdot)$ is absolutely continuous on $[t_0, t_i]$ and for almost all $t \in [t_0, t_i]$

$$\dot{x} \in K[f](x) \tag{2}$$

where $K[f](x) : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is a set-valued map defined as

$$K[f](x) \equiv \bigcap_{\delta > 0} \bigcap_{\mu\{H\} = 0} \overline{\operatorname{co}}\{f(B(x, \delta) \setminus H)\} \tag{3}$$

where $\bigcap_{\mu\{H\}=0}$ denotes the intersection over all sets H of Lebesgue measure zero, $B(x, \delta)$ denotes the ball of radius δ centered at x, $\overline{\operatorname{co}}$ the convex closure and $2^{\mathbb{R}^n}$ the set of subsets of $\mathbb{R}^n$. From now on, it will be assumed, without loss of generality, $t_0 = 0$.

Let us now review the concept of range, norm and boundedness for the set-valued map $K[f](x)$. Further details can be found in [**?**], [**?**]. In particular, let us denote with $R(K[f])$ the range of $K[f]$ defined as

$$R(K[f]) := \bigcup_{x \in \mathbb{R}^n} K[f](x) \tag{4}$$

and with $\|K[f]\|$ the norm of the set-valued map $K[f]$ defined as

$$\|K[f]\| = \sup_{y \in R(K[f])} \|y\| \tag{5}$$

It follows that a set-valued map $K[f]$ is called bounded if its range $R(K[f])$ is bounded, that is if $\|K[f]\| < \infty$.

Briefly, the idea of the Filipov's solution is that the tangent vector to a solution, where it exists, must lie in the convex closure of the values of the vector field in progressively smaller neighborhoods around the solution point. A very important aspect of this definition is given by the possibility of discarding sets of measure zero. Indeed, this technical detail allows solutions to be defined even at points where the vector field itself is not defined.

We now introduce the concept of Clarke's Generalized Gradient, an essential tool in the machinery of nonsmooth analysis.

Definition 2 (Clarke's Generalized Gradient)**.** Consider a locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}$. Then the generalized gradient at x is defined as

$$\partial V(x) = \overline{\operatorname{co}} \left\{ \lim_{i \to \infty} \nabla V(x_i) \,|\, x_i \to x,\, x_i \notin \Omega_V \right\} \tag{6}$$

where Ω_V is the set of measure zeros where the gradient of V is not defined.

We now review the chain rule which allows to differentiate Lipschitz regular functions along the Filipov's solution trajectories.

Theorem 1 (Chain Rule [**?**])**.** *Let $x(\cdot)$ be a Filipov solution to* (1) *on an interval containing t and $V : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz and, in addition, regular function. Then $V(x)$ is absolutely continuous, $(d/dt)V(x(t))$ exists almost everywhere and*

$$\frac{d}{dt} V(x(t)) \in^{a.e.} \mathring{V}(x) \tag{7}$$

where the generalized time derivative $\mathring{V}(x)$ is defined as

$$\mathring{V}(x) := \bigcap_{\xi \in \partial V(x(t))} \xi^T K[f](x) \tag{8}$$

So far, we have introduced the essential tools constituting the machinery of the nonsmooth analysis, where the right-hand side of differential equations may be discontinuous and the Lyapunov function may be non-differentiable. Interestingly, this machinery can be simplified under the assumption of continuous differentiability of the Lyapunov function.

First, let us notice that for a continuously differentiable function the generalized derivative becomes a singleton containing the actual gradient of the function, that is

Corollary 1 ([?]). *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function. Then*

$$\partial V(x) = \{\nabla V(x)\} \tag{9}$$

Then, by exploiting this information we can provide a simplified version of the chain rule stated in Theorem 1 as:

Theorem 2 (Simplified Chain Rule). *Let $x(\cdot)$ be a Filipov solution to (1) on an interval containing t and $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function. Then $V(x)$ is absolutely continuous, $(d/dt)V(x(t))$ exists almost everywhere and*

$$\frac{d}{dt}V(x(t)) \in^{a.e.} \mathring{V}(x) \tag{10}$$

where the generalized time derivative $\mathring{V}(x)$ is defined as

$$\mathring{V}(x) := (\nabla V(x))^T K[f](x) \tag{11}$$

Finally, we review a calculus for computing the Filipov's differential inclusions, originally developed in [?] (and further extended in [?]).

Theorem 3 (Calculus for K [?]). *The map $K : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ has the following properties*

1) *Assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally bounded. Then $\exists\, H_f \subset \mathbb{R}^n$, $\mu\{H_f\} = 0$, such that $\forall\, H \in \mathbb{R}^n$, $\mu\{H\} = 0$,*
$$K[f](x) = \text{co}\left\{\lim_{i\to\infty} f(x_i) \,|\, x_i \to x, x_i \notin H_f \cup H\right\} \tag{12}$$
2) *Assume that $f,\, g : \mathbb{R}^n \to \mathbb{R}^n$ are locally bounded; then*
$$K[f+g](x) \subset K[f](x) + K[g](x) \tag{13}$$
3) *Assume that $f_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$, $j \in \{1, \ldots, N\}$, are locally bounded; then*
$$K\left[\times_{j=1}^N f_j(x)\right] \subset \times_{j=1}^N K[f_j](x) \tag{14}$$
where the cartesian product notation and the column vector notation are used interchangeably.
4) *Let $g : \mathbb{R}^m \to \mathbb{R}^{p\times n}$ (i.e., matrix valued) be continuous and $f : \mathbb{R}^m \to \mathbb{R}^n$ be locally bounded; then*
$$K[g\,f](x) = g(x)\,K[f](x) \tag{15}$$
where $g\,f \triangleq g(x)\,f(x) \in \mathbb{R}^p$.
5) *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous; then*
$$K[f](x) = \{f(x)\} \tag{16}$$
6) *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz; then*
$$K[\nabla f](x) = \partial f(x) \tag{17}$$

IV. Problem Setting

Consider a dynamical system composed of n interacting agents, where each agent i has a non-linear dynamics of the form

$$\dot{x}_i = f(x_i) + A\,x_i + g(u_i) \tag{18}$$

where $A \in \mathbb{R}^{d\times d}$, $f : \mathbb{R}^d \to \mathbb{R}^d$ is a Lipschitz nonlinear functions with Lipschitz constant l_f, and $g : \mathbb{R}^p \to \mathbb{R}^d$ is a measurable and essentially locally bounded function under Assumption 1.

Assumption 1. One of the following conditions must hold:

a) the function $g(\cdot)$ is bounded (with a known upper-bound);
b) the derivative $K[\dot{g}](\cdot)$ is bounded (with a known upper-bound).

Assumption 2. We assume that agent i is aware of its k-hop neighborhood $\mathcal{N}_i^k$.

Note that, this assumption on the awareness of the k-hop neighborhood is reasonable, as the design of distributed neighborhood discovery algorithms is a well-established research topic in the sensor network community (see [?], [?] and reference therein). Furthermore, we point out that the knowledge of the k-hop neighborhood does not necessarily imply also the knowledge of the network topology of the k-hop graph, which is yet another well-established research topic in sensor networks [?], [?].

Let us denote with $\boldsymbol{x}^i$ the vector containing the actual state x_j of the agents belonging to the augmented k-hop neighborhood $\bar{\mathcal{N}}_i^k$ of an agent i, that is

$$\boldsymbol{x}^i = \bar{P}_i\,\boldsymbol{x} = \left[x_{n_i^1}{}^T, \ldots, x_{n_i^{\eta_i}}{}^T\right]^T \tag{19}$$

with $\eta_i = \left|\bar{\mathcal{N}}_i^k\right|$ and matrix P_i is a $\eta_i \times n$ binary matrix with:

$$P_i = \begin{bmatrix} e_{n_i^1} & e_{n_i^2} & \cdots & e_{n_i^{\eta_i}} \end{bmatrix}^{\mathrm{T}} \tag{20}$$

where $e_{n_i^j}$ is the canonical vector with all zeros but element n_i^j which is 1. Finally, let us define the matrix $\bar{P}_i$ as

$$\bar{P}_i = P_i \otimes I_d \tag{21}$$

that is a selection matrix obtained according to the given graph structure $\mathcal{G}$. Note that $\bar{P}_i^T\,\bar{P}_i\,\boldsymbol{x} = \bar{P}_i^T\,\boldsymbol{x}^i$, that is it returns a vector which has the same size as the vector $\boldsymbol{x}$ but it contains only the components of the agents $j \in \bar{\mathcal{N}}_i^k$ by preserving the original index ordering. The state estimate $\hat{\boldsymbol{x}}^i$ of an agent i is defined as

$$\hat{\boldsymbol{x}}^i = \left[\hat{x}_{i,n_i^1}^T, \ldots, \hat{x}_{i,n_i^{\eta_i}}^T\right]^T \tag{22}$$

where $\hat{x}_{i,n_i^p}$ is the estimate of the state $x_{n_i^p}$ carried out by the agent i. Furthermore, let us define the state estimation error for each agent i as

$$\tilde{\boldsymbol{x}}^i = \boldsymbol{x}^i(t) - \hat{\boldsymbol{x}}^i(t). \tag{23}$$

We define the vector containing the actual input $g(u_j)$ of the agents belonging to $\bar{\mathcal{N}}_i^k$ of agent i, $\boldsymbol{g}^i$ as

$$\boldsymbol{g}^i = \left[g(u_{n_i^1})^T, \ldots, g(u_{n_i^{\eta_i}})^T\right]^T = \left[g_{n_i^1}^T, \ldots, g_{n_i^{\eta_i}}{}^T\right]^T \tag{24}$$

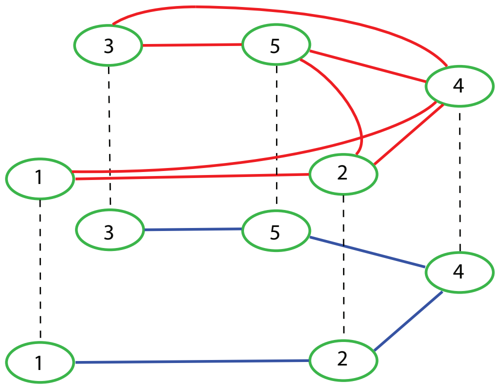


Fig. 1: Example. Network topology where both the real (bottom) and the virtual 2-hop (top) communication graphs are highlighted.

and $\hat{\boldsymbol{g}}^i$ as the corresponding estimate made by agent i itself as

$$\hat{\boldsymbol{g}}^i = \left[\hat{g}_{i,n_i^1}^T, \, \dots, \, \hat{g}_{i,n_i^{\eta_i}}^T\right]^T \tag{25}$$

with $\hat{g}_{i,n_i^j}$ the estimate of $g(u_{n_i^j})$ made by agent i.
Moreover, let $\tilde{\boldsymbol{g}}^i$ be the stacked vector of the estimation errors made by agent i concerning the term $\boldsymbol{g}^i$, i.e.

$$\tilde{\boldsymbol{g}}^i = \boldsymbol{g}^i - \hat{\boldsymbol{g}}^i \tag{26}$$

which will be exploited in the following. The following example is given to better illustrare the role of the matrix $\bar{P}_i$.

Example 1. *Let us consider a network with $n = 5$, $d = 1$ and the communication graph given in Figure 1. We have that the augmented 2-hop ($k_1 = 2$) neighborhood of agent 1 is $\bar{\mathcal{N}}_1^2 = \{1, 2, 4\}$ and the matrix $\bar{P}_1$ is defined as:*

$$\bar{P}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{27}$$

Now, let the global state be $\boldsymbol{x} = [x_1, \, x_2, \, x_3, \, x_4, \, x_5]^T$, it is

$$\boldsymbol{x}^1 = \bar{P}_1 \boldsymbol{x} = [x_1, \, x_2, \, x_4]^T \tag{28}$$

and

$$\bar{P}_1{}^T \boldsymbol{x}^1 = \bar{P}_1{}^T \bar{P}_1 \boldsymbol{x} = [x_1, \, x_2, \, 0, \, x_4, \, 0]^T \tag{29}$$

As far as the neighborhood $\bar{\mathcal{N}}_i^2$ is concerned, for $i = 1$ we have

$$\bar{\mathcal{N}}_1^2 = \{n_1^1, \, n_1^2, \, n_1^3\} \tag{30}$$

with

$$n_1^1 = 1, \quad n_1^2 = 2, \quad n_1^3 = 4 \tag{31}$$

We are now ready to state the problem formulation that we are addressing in this work.

Problem 1. Let us consider a multi-agent system composed of n agents. Our problem is to design a k-hop graph-based local observer for each agent i, for which there exists $T > 0$ such that

$$\|\tilde{\boldsymbol{x}}^i(t)\| = 0, \quad \forall \, t \geq T, \quad i \in \{1, \, \dots, \, n\}. \tag{32}$$

That is each agent i is able to track the state of its k-hop augmented neighborhood $\bar{\mathcal{N}}_i^k$ in finite-time.

V. k-hop Graph-Based State Observer

In this section, we derive a finite-time distributed observer to estimate the state of the agents limiting the interactions to the k-hop neighborhood.

A. Local Observer Definition

Let us assume each agent i updates its state estimate $\hat{\boldsymbol{x}}^i$, defined as in (22), according to the following k-hop graph-based local observer

$$\begin{aligned} \dot{\hat{\boldsymbol{x}}}^i &= \bar{f}(\hat{\boldsymbol{x}}^i) + \bar{A}_i \hat{\boldsymbol{x}}^i + \bar{\Omega}_i \, \bar{G}_i (\boldsymbol{\xi}^i - \hat{\boldsymbol{x}}^i) + \bar{\Theta}_i \, \mathrm{sign}\left(\bar{G}_i \boldsymbol{\xi}^i\right) + \hat{\boldsymbol{g}}^i \\ \boldsymbol{\xi}^i &= \bar{H}_i(\boldsymbol{x}^i - \hat{\boldsymbol{x}}^i) + \sum_{j \in \mathcal{N}_i} \bar{P}_i \left(-\bar{P}_j^T \, \bar{P}_j \bar{P}_i^T \hat{\boldsymbol{x}}^i + \bar{P}_i^T \, \bar{P}_i \bar{P}_j^T \, \hat{\boldsymbol{x}}^j\right) \end{aligned} \tag{33}$$

where $\bar{f}(\hat{\boldsymbol{x}}^i)$ is defined as

$$\bar{f}(\hat{\boldsymbol{x}}^i) = \left[f(\hat{x}_{i,n_i^1})^T, \, \dots, \, f(\hat{x}_{i,n_i^{\eta_i}})^T\right]^T \tag{34}$$

and denotes the estimates of the corresponding quantity

$$\bar{f}(\boldsymbol{x}^i) = \left[f(x_{n_i^1})^T, \, \dots, \, f(x_{n_i^{\eta_i}})^T\right]^T \tag{35}$$

$\bar{\Omega}_i = \Omega_i \otimes I_d$, $\bar{\Theta}_i = \Theta_i \otimes I_d$, with Ω_i and Θ_i diagonal gain matrices defined as:

$$\begin{aligned} \Omega_i &= \mathrm{diag}\{\omega_{n_i^1}, \, \dots, \, \omega_{n_i^{\eta_i}}\} \\ \Theta_i &= \mathrm{diag}\{\theta_{n_i^1}, \, \dots, \, \theta_{n_i^{\eta_i}}\} \end{aligned} \tag{36}$$

with $\omega_p, \theta_p \in \mathbb{R}^+$, and the matrices $\bar{A}_i$, $\bar{G}_i$, $\bar{B}_i$, $\bar{H}_i$ defined as

$$\bar{A}_i = I_{\eta_i} \otimes A, \;\; \bar{B}_i = I_{\eta_i} \otimes B, \;\; \bar{G}_i = I_{\eta_i} \otimes G, \;\; \bar{H}_i = H_i \otimes I_d \tag{37}$$

where the diagonal matrix $H_i \in \mathbb{R}^{\eta_i \times \eta_i}$ is defined as

$$H_i(p, q) = \begin{cases} 1 & \text{if } p = q \wedge n_i^p = i \\ 0 & \text{otherwise} \end{cases} \tag{38}$$

that is, all the diagonal elements are equal to zero but the element corresponding to the agent i itself that is equal to 1. Finally, $G \in \mathbb{R}^{d \times d}$ is a positive symmetric matrix to be designed and the signum function is defined as

$$\mathrm{sign}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ 0 & \text{otherwise} \end{cases} \tag{39}$$

Regarding the term ξ_i, we can notice that is it composed of three contributions:

- $\bar{H}_i(\boldsymbol{x}^i - \hat{\boldsymbol{x}}^i)$: this term is local since x_i is known by agent i and is used to drive $\hat{x}_{i,i} = \bar{H}_i \hat{\boldsymbol{x}}^i$ to $x_i = \bar{H}_i \boldsymbol{x}^i$;
- $\sum_{j \in \mathcal{N}_i} \bar{P}_i \left(-\bar{P}_j^T \, \bar{P}_j \bar{P}_i^T \hat{\boldsymbol{x}}^i + \bar{P}_i^T \, \bar{P}_i \bar{P}_j^T \, \hat{\boldsymbol{x}}^j\right)$: this standard consensus-like term compares the estimates made by agent i and by its k-hop neighbors
- $\hat{\boldsymbol{x}}^i$: this term is introduced in order to have eq. (33) in a form that would recall the classical innovation term in the observer design.

Remark 1. Note that, in order to build the matrix $\bar{P}_j$, it is required the knowledge of the k-hop neighborhood of agent j; this could induce to believe that when performing $\bar{P}_i \bar{P}_j^T$ in the observer dynamics of the i-th agent, such an agent should posses knowledge concerning the k-hop neighborhood of agent j. As a matter of fact, this mathematical notation, even though very useful for the theoretical analysis, does not reflect the actual requirements for the implementation of the observer. In practice, in order to run the algorithm, under the assumption of global indexes for the agents, it suffices that agent j shares the vector of estimates with the related global indexes and agent i can locally filter out all the information concerning agents which do not belong to its k-hop neighborhood (which is known according to Assumption 2). We highlight that the requirement of having global indexes is not demanding especially when dealing with communication networks where, for example, the MAC address could be used for this purpose.

B. Local Observer Theoretical Properties

In this section, we are focused on demonstrating that the k-hop local observer given in eq. (33) solves Problem 1. In doing so, rather than proving directly eq. (32), we will proceed by changing point of view. That is, instead of proving that each agent i is able to estimate the state of the k-hop neighboring agents (namely, $\boldsymbol{x}_i$), we will first demonstrate that this requirement is equivalent to show that the local state x_i of each agent i is estimated by the agents belonging to its k-hop augmented neighborhood $\bar{\mathcal{N}}_i^k$; and, then, we will prove that this is achieved by the observer given in eq. (33).

In this regard, let us consider the generic agent i and let us notice that for any index $l \in \bar{\mathcal{N}}_i^k$, we have $\hat{x}_{i,l} = S_l \bar{P}_i^T \hat{\boldsymbol{x}}^i$, while for any index $l \notin \bar{\mathcal{N}}_i^k$ we have that $S_l \bar{P}_l^T \hat{\boldsymbol{x}}^i$ is zero with S_l a selection matrix defined as

$$S_l = \{O_d, \ldots, \underbrace{I_d}_{l-th}, \ldots, \underbrace{O_d}_{n-th}\}$$

At this point, let us define $\hat{\boldsymbol{\zeta}}^i$ as the vector that collects the estimates of the state x_i computed by all agents j having agent i in their k-hop neighborhood:

$$\hat{\boldsymbol{\zeta}}^i = \begin{bmatrix} S_i \bar{P}_{n_i^1}^T \hat{\boldsymbol{x}}^{n_i^1} \\ \vdots \\ S_i \bar{P}_{n_i^{\eta_i}}^T \hat{\boldsymbol{x}}^{n_i^{\eta_i}} \end{bmatrix} = \begin{bmatrix} \hat{x}_{n_i^1,i} \\ \vdots \\ \hat{x}_{n_i^{\eta_i},i} \end{bmatrix} \tag{40}$$

where $\hat{x}_{n_i^p,i}$ is the estimate of the state x_i carried out by the agent n_i^p. In addition, let us define the vector $\boldsymbol{\zeta}^i$ collecting η_i copies of the state x_i as

$$\boldsymbol{\zeta}^i = 1_{\eta_i} \otimes x_i \tag{41}$$

Furthermore, let us denote with $\tilde{\boldsymbol{\zeta}}^i$ the collection of the estimation errors of its state x_i, made by the agents belonging to $\mathcal{V}_i$

$$\tilde{\boldsymbol{\zeta}}^i = \boldsymbol{\zeta}^i - \hat{\boldsymbol{\zeta}}^i \tag{42}$$

We now provide our first result, which states an equivalence between the fact an agent i can estimate the state of the agents belonging to the augmented k-hop neighborhood $\bar{\mathcal{N}}_i^k$, and the fact that the state x_i of an agent i can be estimated by the agents belonging to its k-hop augmented neighborhood $\bar{\mathcal{N}}_i^k$.

Lemma 1. *Consider a multi-agent system running the local k-hop observer given in eq.* (33)*. The following facts are equivalent:*

- $\| \tilde{\boldsymbol{\zeta}}^i \| \rightarrow 0, \quad \forall\, i \in \{1, \ldots, n\}$
- $\| \tilde{\boldsymbol{x}}^i \| \rightarrow 0, \quad \forall\, i \in \{1, \ldots, n\}$

Proof. The proof simply follows from the observation that by considering the stacked vectors

$$\tilde{\boldsymbol{\zeta}} = \begin{bmatrix} \tilde{\boldsymbol{\zeta}}^1 \\ \vdots \\ \tilde{\boldsymbol{\zeta}}^n \end{bmatrix}, \quad \tilde{\boldsymbol{x}} = \begin{bmatrix} \tilde{\boldsymbol{x}}^1 \\ \vdots \\ \tilde{\boldsymbol{x}}^n \end{bmatrix}$$

with $i \in (1, \ldots, n)$, it is always possible for any graph L and for each k to find a permutation matrix T of proper dimensions, that is full-rank by construction, such that $\tilde{\boldsymbol{x}} = T \tilde{\boldsymbol{\zeta}}$. □

From Lemma 1 it follows that, from a mathematical standpoint, Problem 1, and more precisely eq. (32), can be equivalently restated according to the following formulation.

Problem 2. Let us consider a multi-agent system composed of n agents. Our problem is to design a k-hop graph-based local observer for each agent i, for which there exists $T_{x,i} > 0$ such that

$$\|\tilde{\boldsymbol{\zeta}}^i\| = 0, \ \ \forall\, t \geq T_{x,i}, \ \ i \in \{1, \ldots, n\} \tag{43}$$

In the rest of the section, we focus our attention on demonstrating that the condition given in eq. (43) can be proven by letting each agent i run the k-hop local observer given in eq. (33). Let us now introduce the (sub)-graph $\mathcal{G}_i$ obtained by reducing $\mathcal{G}$ with respect to $\mathcal{V}_i$, that is $\mathcal{G}_i = \{\mathcal{V}_i, \mathcal{E}_i\}$ where $\mathcal{V}_i$ and $\mathcal{E}_i \subseteq \mathcal{E} \,:\, (p,q) \in \mathcal{E}_i \leftrightarrow \{p, q\} \in \mathcal{V}_i$. At this point, we can define the Laplacian matrix L_i associated to $\mathcal{G}_i$ as

$$L_i = D_i - A_i \tag{44}$$

where D_i and A_i are the related degree matrix and adjacency matrix, respectively. Let us now define the matrix M_i as

$$M_i = L_i + H_i \tag{45}$$

The following result shows that the matrix M_i is positive definite.

Lemma 2. *Consider M_i as in eq.* (45)*, then in the case of connected undirected graphs the following holds*

$$\lambda_i > 0, \quad \forall\, \lambda_i \in \sigma\left(M_i\right) \tag{46}$$

Proof. The proof follows the same steps as in [**?**]; then, it is here omitted for the sake of brevity. Intuitively, the result follows from the connectedness of the Laplacian matrix L_i associated to (sub)-graph $\mathcal{G}_i$. □

The following lemma provides the form of the error dynamics $\tilde{\boldsymbol{\zeta}}^i$ for all $i \in \{1, \ldots, n\}$. To this end, let us first denote with $\boldsymbol{\varrho}^i$ the vector collecting η_i copies of the input $g(u_i)$ as

$$\boldsymbol{\varrho}^i = 1_{\eta_i} \otimes g(u_i) \tag{47}$$

and with $\hat{\boldsymbol{\varrho}}^i$ the corresponding estimate which collects, as usual, the estimates of $g(u_i)$ made by the agent n_i^p with $p \in \mathcal{V}_i$, i.e.

$$\hat{\boldsymbol{\varrho}}^i = \left[\hat{g}_{i,n_i^1}^T, \ldots, \hat{g}_{i,n_i^{\eta_i}}^T\right]^T \tag{48}$$

and we use $\tilde{\boldsymbol{\varrho}}^i$ to denote the quantity

$$\tilde{\boldsymbol{\varrho}}^i = \boldsymbol{\varrho}^i - \hat{\boldsymbol{\varrho}}^i \tag{49}$$

Furthermore, let us denote with $\bar{f}(\boldsymbol{\zeta}^i)$ the vector

$$\bar{f}(\hat{\boldsymbol{\zeta}}^i) = \left[f(\hat{x}_{n_i^1,i})^T, \ldots, f(\hat{x}_{n_i^{\eta_i},i})^T\right]^T. \tag{50}$$

Lemma 3. *Consider a multi-agent system running the local k-hop observer given in eq.* (33). *The error dynamics of $\tilde{\boldsymbol{\zeta}}^i$ with $i \in \{1, \dots, n\}$ has the following form:*

$$\begin{aligned}\dot{\tilde{\boldsymbol{\zeta}}}^i = &(\bar{f}(\boldsymbol{\zeta}^i) - \bar{f}(\hat{\boldsymbol{\zeta}}^i)) + \left(\bar{A}_i - \omega_i(M_i \otimes G\right) \tilde{\boldsymbol{\zeta}}^i \\ &-\theta_i \operatorname{sign}\left((M_i \otimes G)\tilde{\boldsymbol{\zeta}}^i\right) + \tilde{\boldsymbol{\varrho}}^i\end{aligned} \tag{51}$$

Proof. The dynamics of the generic term $\hat{x}_{l,i}$ in (40) can be expressed as

$$\begin{aligned}\dot{\hat{x}}_{l,i} &= S_i \bar{P}_l^T \dot{\hat{\boldsymbol{x}}}^l \\ &= S_i \bar{P}_l^T \left(\bar{f}(\hat{\boldsymbol{x}}^l) + \bar{A}_l \hat{\boldsymbol{x}}^l + \omega_i \bar{G}_l \boldsymbol{\xi}^l \right. \\ &\left. + \theta_i \operatorname{sign}\left(\bar{G}_l \boldsymbol{\xi}^l\right) + \hat{\boldsymbol{g}}^l\right) \\ &= f(\hat{x}_{l,i}) + A\hat{x}_{l,i} + \omega_i G S_i \bar{P}_l^T \boldsymbol{\xi}^l \\ &+ \theta_i \operatorname{sign}\left(G S_i \bar{P}_l^T \boldsymbol{\xi}^l\right) + \hat{g}_{l,i}\end{aligned} \tag{52}$$

Let us now detail the form of the term $S_i \bar{P}_l^T \boldsymbol{\xi}^l$. In particular, it should be noticed that this term is identically zero for $l \notin \mathcal{V}_i$; differently, for each $l \in \mathcal{V}_i$, it takes the following form

$$\begin{aligned}S_i \bar{P}_l^T \boldsymbol{\xi}^l &= S_i \bar{P}_l^T \bar{H}_l (\boldsymbol{x}^l - \hat{\boldsymbol{x}}^l) \\ &+ S_i \bar{P}_l^T \sum_{j \in \mathcal{N}_l} \bar{P}_l \Big(\bar{P}_l^T \bar{P}_l \bar{P}_j^T \hat{\boldsymbol{x}}^j - \bar{P}_j^T \bar{P}_j \bar{P}_l^T \hat{\boldsymbol{x}}^l\Big) \\ &= S_i \bar{P}_l^T \bar{H}_l (\boldsymbol{x}^l - \hat{\boldsymbol{x}}^l) + \sum_{j \in \mathcal{N}_l \cap \mathcal{V}_i} (\hat{x}_{l,i} - \hat{x}_{j,i})\end{aligned} \tag{53}$$

This expression can be further simplified by noticing that for $l = i$ we have $S_i \bar{P}_l^T \bar{H}_l \boldsymbol{x}^l = x_i$, and $\mathcal{N}_l \cap \mathcal{V}_i = \mathcal{N}_i$, from which it follows

$$S_i \bar{P}_i^T \boldsymbol{\xi}^i = (x_i - \hat{x}_{i,i}) + \sum_{j \in \mathcal{N}_i} (\hat{x}_{i,i} - \hat{x}_{j,i}) \tag{54}$$

and for $l \neq i$ we have $S_i \bar{P}_l^T \bar{H}_l \boldsymbol{x}_l = 0_d$, from which it follows

$$S_i \bar{P}_l^T \boldsymbol{\xi}^l = \sum_{j \in \mathcal{N}_l \cap \mathcal{V}_i} (\hat{x}_{l,i} - \hat{x}_{j,i}) \tag{55}$$

where it should be noticed that $l \in \mathcal{N}_i^k$, and $\mathcal{N}_l \cap \mathcal{V}_i$ is different from the empty set by construction.

Summarizing, eq. (52) for $l = i$ can be restated as

$$\begin{aligned}\dot{\hat{x}}_{i,i} &= f(\hat{x}_{i,i}) + A\hat{x}_{i,i} + \omega_i G \left((x_i - \hat{x}_{i,i}) + \sum_{j \in \mathcal{N}_i} (\hat{x}_{j,i} - \hat{x}_{i,i})\right) \\ &+ \theta_i \operatorname{sign}\left[G\left((x_i - \hat{x}_{i,i}) + \sum_{j \in \mathcal{N}_i} (\hat{x}_{j,i} - \hat{x}_{i,i})\right)\right] + \hat{g}_{l,i}\end{aligned} \tag{56}$$

and for $l \neq i$ can be restated as

$$\begin{aligned}\dot{\hat{x}}_{l,i} &= f(\hat{x}_{l,i}) + A\hat{x}_{l,i} + \omega_i G \left(\sum_{j \in \mathcal{N}_l \cap \mathcal{N}_i^k} (\hat{x}_{j,i} - \hat{x}_{l,i})\right) \\ &+ \theta_i \operatorname{sign}\left[G\left(\sum_{j \in \mathcal{N}_l \cap \mathcal{V}_i} (\hat{x}_{j,i} - \hat{x}_{l,i})\right)\right] + \hat{g}_{l,i}\end{aligned} \tag{57}$$

At this point, based on eqs. (56) and (57), we can write the dynamics of the stacked vector $\hat{\boldsymbol{\zeta}}^i$ defined in eq. (40) collecting the estimate of the state x_i carried out by the agents $j \in \mathcal{V}_i$ as

$$\begin{aligned}\dot{\hat{\boldsymbol{\zeta}}}^i &= \bar{f}(\hat{\boldsymbol{\zeta}}^i) + \bar{A}_i \hat{\boldsymbol{\zeta}}^i + \omega_i \bar{G}_i \left(\bar{L}_i + \bar{H}_i\right) \tilde{\boldsymbol{\zeta}}^i \\ &+ \theta_i \operatorname{sign}\left(\bar{G}_i (\bar{L}_i + \bar{H}_i) \tilde{\boldsymbol{\zeta}}^i\right) + \hat{\boldsymbol{\varrho}}^i\end{aligned} \tag{58}$$

where $\bar{L}_i = L_i \otimes I_d$ and the following fact has been exploited

$$\bar{L}_i \boldsymbol{\zeta}^i = (L_i \otimes I_d)(1_{\eta_i} \otimes x_i) = (L_i 1_{\eta_i}) \otimes (I_d x_i) = 0_{\eta_i d} \tag{59}$$

At this point, by noticing that the dynamics of $\boldsymbol{\zeta}^i$ can be written as

$$\dot{\boldsymbol{\zeta}}^i = \bar{f}(\boldsymbol{\zeta}^i) + \bar{A}_i \boldsymbol{\zeta}^i + \boldsymbol{\varrho}^i \tag{60}$$

it follows that the error dynamics of stacked vector $\dot{\tilde{\boldsymbol{\zeta}}}^i$ is

$$\begin{aligned}\dot{\tilde{\boldsymbol{\zeta}}}^i = &(\bar{f}(\boldsymbol{\zeta}^i) - \bar{f}(\hat{\boldsymbol{\zeta}}^i)) + \left(\bar{A}_i - \omega_i \bar{G}_i (\bar{L}_i + \bar{H}_i)\right) \tilde{\boldsymbol{\zeta}}^i \\ &-\theta_i \operatorname{sign}\left(\bar{G}_i (\bar{L}_i + \bar{H}_i) \tilde{\boldsymbol{\zeta}}^i\right) + \tilde{\boldsymbol{\varrho}}^i\end{aligned} \tag{61}$$

Finally, by resorting to well-known properties of the Kronecker product, the following holds

$$\bar{G}_i(\bar{L}_i + \bar{H}_i) = (L_i + H_i) \otimes G \tag{62}$$

which allows to rewrite eq. (61) as

$$\begin{aligned}\dot{\tilde{\boldsymbol{\zeta}}}^i = &(\bar{f}(\boldsymbol{\zeta}^i) - \bar{f}(\hat{\boldsymbol{\zeta}}^i)) + \left(\bar{A}_i - \omega_i (M_i \otimes G)\right) \tilde{\boldsymbol{\zeta}}^i \\ &-\theta_i \operatorname{sign}\left((M_i \otimes G)\tilde{\boldsymbol{\zeta}}^i\right) + \tilde{\boldsymbol{\varrho}}^i\end{aligned} \tag{63}$$

thus completing the proof. □

The following lemma provides a set of conditions which will be used later to establish the convergence of the error dynamics given in eq. (51).

Lemma 4. *Let us consider $\bar{A}_i$ and M_i defined as in eqs.* (37) *and* (45)*, respectively. Then, the following holds true*

$$(M_i \otimes G)\left(\bar{A}_i - \omega_i (M_i \otimes G)\right) + l_f \|(M_i \otimes G)\| I_{\eta_i d} \prec 0 \tag{64}$$

if ω_i and G are chosen such that

$$\omega_i > \frac{1}{\lambda_j}\left(1 + \frac{l_f \|(M_i \otimes G)\|}{\lambda_j \lambda_{\min}(G^{\mathrm{T}} G)}\right) \tag{65}$$

and

$$G^{\mathrm{T}} A + A^{\mathrm{T}} G - 2G^{\mathrm{T}} G \prec 0. \tag{66}$$

Proof. In order to prove the lemma, we notice that M_i is a symmetric positive definite matrix and, thus, there exists a matrix $T \in \mathbb{R}^{\eta_i \times \eta_i}$ such that $T \Lambda_i T^T = M_i$, with Λ_i a diagonal matrix. Now, by exploiting the properties of the Kronecker product we have

$$\begin{aligned}&(M_i \otimes G)\left(\bar{A}_i - \omega_i (M_i \otimes G)\right) = \\ &\underbrace{\left(T\Lambda_i T^T \otimes G\right) \bar{A}_i}_{T_1} - \underbrace{\omega_i \left(T\Lambda_i T^T \otimes G\right)\left(T\Lambda_i T^T \otimes G\right)}_{T_2}\end{aligned} \tag{67}$$

By recalling the property

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD) \tag{68}$$

and that $\bar{A}_i = I_{\eta_i} \otimes A$, we have

$$\bar{A}_i = I_{\eta_i} \otimes A = (T_i \otimes I)\,(I_{\eta_i} \otimes A)\left(T_i^T \otimes I\right) \tag{69}$$

and

$$\left(T_i \Lambda_i T_i^T \otimes G\right) = (T_i \otimes I)\,(\Lambda_i \otimes G)\left(T_i^T \otimes I\right) \tag{70}$$

At this point, by exploiting eqs. (69) and (70) and by taking into account the property

$$T_i A T_i^T \otimes T_i B T_i^T = T_i (A \otimes B) T_i^T \tag{71}$$

we can write the first term T_1 as

$$(T_i \otimes I)\left[(\Lambda_i \otimes G)\,(I_{\eta_i} \otimes A)\right]\left(T_i^T \otimes I\right) \tag{72}$$

and the second term T_2 as

$$\omega_i\ (T_i \otimes I)\left[(\Lambda_i \otimes G)\,(\Lambda_i \otimes G)\right]\left(T_i^T \otimes I\right) \tag{73}$$

Furthermore, by substituting eqs. (72) and (73) in eq. (67) we obtain

$$(T_i \otimes I)\left[(\Lambda_i \otimes G)\left((I_{\eta_i} \otimes A) - \omega_i\,(\Lambda_i \otimes G)\right)\right]\left(T_i^T \otimes I\right) \tag{74}$$

from which it can be noticed that the negative definiteness of the matrix $(M_i \otimes G)\left(\bar{A}_i - \omega_i (M_i \otimes G)\right)$ can be equivalently stated in terms of the matrix $(\Lambda_i \otimes G)\left((I_{\eta_i} \otimes A) - \omega_i\,(\Lambda_i \otimes G)\right)$. By focusing on this last matrix, we notice that

$$\begin{aligned}&(\Lambda_i \otimes G)\left((I_{\eta_i} \otimes A) - \omega_i\,(\Lambda_i \otimes G)\right) = \\ &(\Lambda_i \otimes G\,A) - \omega_i\left({\Lambda_i}^2 \otimes G^T G\right)\end{aligned} \tag{75}$$

Now, in order to have

$$(\Lambda_i \otimes G\,A) - \omega_i\left({\Lambda_i}^2 \otimes G^T G\right) + l_f \|(M_i \otimes G)\| I_{\eta_i d} \prec 0 \tag{76}$$

since Λ_i is a diagonal matrix, the result of the Kronecker product is a block-diagonal matrix, for which the matrices on the main diagonal have the following structure

$$\lambda_j\left(G^{\mathrm{T}} A - \omega_i\,\lambda_j\,G^T G\right) + l_f \|(M_i \otimes G)\| I_d, \quad \lambda_j \in \sigma\left(M_i\right) \tag{77}$$

which can be equivalently written as:

$$\lambda_j\left(G^{\mathrm{T}} A - \omega_i\,\lambda_j\,G^T G + \frac{l_f\,\|(M_i \otimes G)\|}{\lambda_j} I_d\right) \tag{78}$$

By recalling from Lemma 2 that $\lambda_j > 0$, under the assumption

$$\omega_i > \frac{1}{\lambda_j}\left(1 + \frac{l_f\,\|(M_i \otimes G)\|}{\lambda_j \lambda_{\min}(G^{\mathrm{T}} G)}\right) \tag{79}$$

we have that if the following holds true

$$\left(G^{\mathrm{T}} A - G^T G\right) \prec 0 \tag{80}$$

then, by construction also the following holds

$$\left(G^{\mathrm{T}} A - \omega_i\,\lambda_j\,G^T G + \frac{l_f\,\|(M \otimes G)\|}{\lambda_j} I_d\right) \prec 0 \tag{81}$$

Finally, we notice that the negative definiteness of the matrix $G\,A - G^T G$ follows from the negative definiteness of its symmetric part, that is

$$\frac{1}{2}\left(G^{\mathrm{T}} A + A^T G - 2 G^T G\right) \prec 0 \tag{82}$$

where it should be noticed that the existence of a matrix G that satisfies eq. (66) is ensured by the fact that (A, I_n) is stabilizable and observable [?, Theorem 2]. Thus, the result follows. □

Remark 2. The tuning of the gain ω_i in eq. (79) implies the knowledge of the minimum eigenvalue of L_i, where the underlying graph $\mathcal{G}_i$ scales only according to the k-hop neighborhood of agent i. However, as pointed out in [?] and references therein, this information is generally not available to the agents. Nevertheless, it can be estimated for example, by resorting to adaptive control techniques [?] or by performing spectral estimation analysis [?].

We are now ready to establish our main result, that is a set of conditions under which the convergence of the error dynamics given in eq. (51) can be guaranteed. In doing so, we recall that, due to the equivalence established in Lemma 1, this also guarantees the convergence of the proposed k-hop observer given in eq. (33). Thus providing a solution for Problem 1.

Theorem 4. *For $i \in \{1, \ldots, n\}$, let us consider the error dynamics given in eq. (51) and let us assume that $\|K[\tilde{\boldsymbol{\varrho}}^i]\|$ is bounded in the sense defined in Section III-B. Then, $\tilde{\boldsymbol{\zeta}}^i$ reaches the origin in finite time $T_{x,i}$*

$$T_{x,i} \leq \sqrt{2}\frac{\lambda_{\max}(M_i)\lambda_{\max}(G)}{\phi_i}\tilde{\boldsymbol{\zeta}}^i(0)$$

given that the gain θ_i satisfies

$$\theta_i > \frac{\lambda_{\max}(M_i)\,\lambda_{\max}(G)}{\lambda_{\min}(M_i)\,\lambda_{\min}(G)}\,\|K[\tilde{\boldsymbol{\varrho}}^i]\| \tag{83}$$

and that the condition in eq. (64) of Lemma 4 holds true.

Proof. To prove the result we notice that the proposed distributed finite-time state observer and the resulting error dynamics are discontinuous thus the nonsmooth analysis must be used. Consider the following continuous differentiable function as Lyapunov candidate function

$$V_i(\tilde{\boldsymbol{\zeta}}^i) = \frac{1}{2}\tilde{\boldsymbol{\zeta}}^{iT}\,(M_i \otimes G)\;\tilde{\boldsymbol{\zeta}}^i \tag{84}$$

To compute the generalized time derivative $\mathring{V}_i(\tilde{\boldsymbol{\zeta}}^i)$ we notice that $V_i(\tilde{\boldsymbol{\zeta}}^i)$ is a continuous differentiable function and that the right-hand side of eq. (51) is discontinuous, thus Theorem 2 applies, that is

$$\begin{aligned}\mathring{V}_i(\tilde{\boldsymbol{\zeta}}^i) &= \left(\nabla V_i(\tilde{\boldsymbol{\zeta}}^i)\right)^T K\left[\dot{\tilde{\boldsymbol{\zeta}}}^i\right] \\ &= \tilde{\boldsymbol{\zeta}}^{iT}\,(M_i \otimes G)\;K\left[\dot{\tilde{\boldsymbol{\zeta}}}^i\right]\end{aligned} \tag{85}$$

At this point, by exploiting eq. (51) and the sum property of the Filippov calculus given in Theorem 3 we have

$$\begin{aligned}\mathring{V}_i(\tilde{\boldsymbol{\zeta}}^i) \subset\ & \tilde{\boldsymbol{\zeta}}^{iT}\,(M_i \otimes G)\,K\left[\bar{f}(\boldsymbol{\zeta}^i) - \bar{f}(\hat{\boldsymbol{\zeta}}^i)\right] \\ &+ \tilde{\boldsymbol{\zeta}}^{iT}\,(M_i \otimes G)\,K\left[\left(\bar{A}_i - \omega_i (M_i \otimes G)\right)\tilde{\boldsymbol{\zeta}}^i\right] \\ &+ \tilde{\boldsymbol{\zeta}}^{iT}\,(M_i \otimes G)\,K\left[-\theta_i\,\mathrm{sign}\left((M_i \otimes G)\tilde{\boldsymbol{\zeta}}^i\right)\right] \\ &+ \tilde{\boldsymbol{\zeta}}^{iT}\,(M_i \otimes G)\,K\left[\tilde{\boldsymbol{\varrho}}^i\right]\end{aligned} \tag{86}$$

In particular, by exploiting the continuity property of the Filippov calculus given in Theorem 3, we have that the following holds for the first and second term of eq. (86)

$$\tilde{\boldsymbol{\zeta}}^{iT}(M_i \otimes G) K\left[\bar{f}(\boldsymbol{\zeta}^i) - \bar{f}(\hat{\boldsymbol{\zeta}}^i)\right] = \tilde{\boldsymbol{\zeta}}^{iT}(M_i \otimes G)(\bar{f}(\boldsymbol{\zeta}^i) - \bar{f}(\hat{\boldsymbol{\zeta}}^i)) \tag{87}$$

and

$$\tilde{\boldsymbol{\zeta}}^{iT}(M_i \otimes G) K\left[\left(\bar{A}_i - \omega_i(M_i \otimes G)\right)\tilde{\boldsymbol{\zeta}}^i\right] = \tilde{\boldsymbol{\zeta}}^{iT}(M_i \otimes G)\left(\bar{A}_i - \omega_i(M_i \otimes G)\right)\tilde{\boldsymbol{\zeta}}^i \tag{88}$$

Furthermore, by exploiting again the multiplication property of the Filippov calculus given in Theorem 3 we have that the following holds for the third term of eq. (86)

$$\begin{aligned}\tilde{\boldsymbol{\zeta}}^{iT}(M_i \otimes G) K\left[-\theta_i \operatorname{sign}\left((M_i \otimes G)\tilde{\boldsymbol{\zeta}}^i\right)\right] &= \\ -\theta_i \tilde{\boldsymbol{\zeta}}^{iT}(M_i \otimes G) K\left[\operatorname{sign}\left((M_i \otimes G)\tilde{\boldsymbol{\zeta}}^i\right)\right] &= \\ -\theta_i \|(M_i \otimes G)\tilde{\boldsymbol{\zeta}}^i\|_1\end{aligned} \tag{89}$$

where it should be notice that

$$K[\operatorname{sign}(x)] \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ [-1,\, 1] & \text{if } x = 0 \end{cases} \tag{90}$$

and that by construction it holds

$$x^{\mathrm{T}} K[\operatorname{sign}(x)] = \begin{cases} \|x\|_1 & \text{if } x > 0 \\ \|x\|_1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases} = \|x\|_1 \tag{91}$$

At this point, by exploiting eqs. (87), (88), (89), it follows that eq. (86) can be re-written as

$$\begin{aligned}\mathring{V}_i(\tilde{\boldsymbol{\zeta}}^i) &= \tilde{\boldsymbol{\zeta}}^{iT}(M_i \otimes G)(\bar{f}(\boldsymbol{\zeta}^i) - \bar{f}(\hat{\boldsymbol{\zeta}}^i)) \\ &+ \tilde{\boldsymbol{\zeta}}^{iT}(M_i \otimes G)\left(\bar{A}_i - \omega_i(M_i \otimes G)\right)\tilde{\boldsymbol{\zeta}}^i \\ &- \theta_i \|(M_i \otimes G)\tilde{\boldsymbol{\zeta}}^i\|_1 + \tilde{\boldsymbol{\zeta}}^{iT}(M_i \otimes G)\ K[\tilde{\boldsymbol{\varrho}}^i]\end{aligned} \tag{92}$$

and by exploiting the notion of norm for set-valued maps introduced in Section (III) we have

$$\begin{aligned}\mathring{V}_i(\tilde{\boldsymbol{\zeta}}^i) &\leq \|\tilde{\boldsymbol{\zeta}}^i\|\|(M_i \otimes G)\|\|\bar{f}(\boldsymbol{\zeta}^i) - \bar{f}(\hat{\boldsymbol{\zeta}}^i)\| \\ &+ \tilde{\boldsymbol{\zeta}}^{iT}(M_i \otimes G)\left(\bar{A}_i - \omega_i(M_i \otimes G)\right)\tilde{\boldsymbol{\zeta}}^i \\ &- \theta_i \|\tilde{\boldsymbol{\zeta}}^{iT}(M_i \otimes G)\|_1 + \\ &+ \|\tilde{\boldsymbol{\zeta}}^i\|\|(M_i \otimes G)\|\|K[\tilde{\boldsymbol{\varrho}}^i]\|\end{aligned} \tag{93}$$

Concerning the first term of the right-hand side of eq. (93), it holds

$$\begin{aligned}&\|\tilde{\boldsymbol{\zeta}}^i\|\|(M_i \otimes G)\|\|\bar{f}(\boldsymbol{\zeta}^i) - \bar{f}(\hat{\boldsymbol{\zeta}}^i)\| \leq l_f \|(M_i \otimes G)\|\|\tilde{\boldsymbol{\zeta}}^i\|^2 \\ &= \tilde{\boldsymbol{\zeta}}^{iT}\left[l_f \|(M_i \otimes G)\| I_d\right]\tilde{\boldsymbol{\zeta}}^i\end{aligned} \tag{94}$$

that, together with the second term and by exploiting Lemma 4, leads to

$$\begin{aligned}&\tilde{\boldsymbol{\zeta}}^{iT}(M_i \otimes G)\left(\bar{A}_i - \omega_i(M_i \otimes G)\right)\tilde{\boldsymbol{\zeta}}^i \\ &+ \|\tilde{\boldsymbol{\zeta}}^i\|\|(M_i \otimes G)\|\|\bar{f}(\boldsymbol{\zeta}^i) - \bar{f}(\hat{\boldsymbol{\zeta}}^i)\| \\ &\leq \tilde{\boldsymbol{\zeta}}^{iT}\left[(M_i \otimes G)\left(\bar{A}_i - \omega_i(M_i \otimes G)\right)\right. \\ &+ \left.(l_f \|(M_i \otimes G)\| I_d)\right]\tilde{\boldsymbol{\zeta}}^i \leq 0\end{aligned} \tag{95}$$

Therefore, eq. (93) can be finally rewritten as

$$\begin{aligned}\mathring{V}_i(\tilde{\boldsymbol{\zeta}}^i) &\leq -\theta_i \|\tilde{\boldsymbol{\zeta}}^{iT}(M_i \otimes G)\|_1 + \|\tilde{\boldsymbol{\zeta}}^i\|\|(M_i \otimes G)\|\|K[\tilde{\boldsymbol{\varrho}}^i]\| \\ &\leq -\theta_i \|(M_i \otimes G)\tilde{\boldsymbol{\zeta}}^i\| + \|\tilde{\boldsymbol{\zeta}}^i\|\|(M_i \otimes G)\|\|K[\tilde{\boldsymbol{\varrho}}^i]\|\end{aligned} \tag{96}$$

At this point, by considering that

$$\begin{aligned}\|(M_i \otimes G)\tilde{\boldsymbol{\zeta}}^i\|_1 \geq \|(M_i \otimes G)\tilde{\boldsymbol{\zeta}}^i\| &\geq \|(M_i \otimes G)\|\|\tilde{\boldsymbol{\zeta}}^i\| \\ &\geq \lambda_{\min}(M_i \otimes G)\|\tilde{\boldsymbol{\zeta}}^i\| \\ &\geq \lambda_{\min}(M_i)\lambda_{\min}(G)\|\tilde{\boldsymbol{\zeta}}^i\|\end{aligned} \tag{97}$$

the following holds for the generalized Lyapunov derivative

$$\begin{aligned}\mathring{V}_i(\tilde{\boldsymbol{\zeta}}^i) &\leq -\theta_i \lambda_{\min}(M_i)\lambda_{\min}(G)\|\tilde{\boldsymbol{\zeta}}^i\| \\ &+ \|\tilde{\boldsymbol{\zeta}}^i\|\|(M_i \otimes G)\|\|K[\tilde{\boldsymbol{\varrho}}^i]\| \\ &\leq -\phi_i \|\tilde{\boldsymbol{\zeta}}^i\|\end{aligned} \tag{98}$$

with ϕ_i defined as

$$\phi_i = \left[\theta_i \lambda_{\min}(M_i)\lambda_{\min}(G) - \|(M_i \otimes G)\|\|K\left[\tilde{\boldsymbol{\varrho}}^i\right]\|\right] \tag{99}$$

Now, by recalling the definition of the Lyapunov function as in eq. (84), we have

$$V_i(\tilde{\boldsymbol{\zeta}}^i)^{\frac{1}{2}} \leq \sqrt{\lambda_{\max}(M_i)\lambda_{\max}(G)}\|\tilde{\boldsymbol{\zeta}}^i\| \tag{100}$$

and thus for the generalized derivative $\mathring{V}_i(\tilde{\boldsymbol{\zeta}}^i)$ the following holds

$$\mathring{V}_i(\tilde{\boldsymbol{\zeta}}^i) \leq -V_i(\tilde{\boldsymbol{\zeta}}^i)^{\frac{1}{2}} \frac{\phi_i}{\sqrt{\lambda_{\max}(M_i)\lambda_{\max}(G)}} \tag{101}$$

At this point, by noticing from Theorem 2 that $\dot{V}(\tilde{\boldsymbol{\zeta}}^i) \in^{\text{a.e.}} \mathring{V}_i(\tilde{\boldsymbol{\zeta}}^i)$, the finite-time convergence of the Lyapunov function to the origin can be attained by choosing θ_i such that $\phi_i > 0$, that is

$$\theta_i > \frac{\lambda_{\max}(M_i)\lambda_{\max}(G)}{\lambda_{\min}(M_i)\lambda_{\min}(G)}\|\bar{B}_i\|\|K[\tilde{\boldsymbol{\varrho}}^i]\| \tag{102}$$

and the setting time is

$$\begin{aligned}T_{x,i} &= 2\frac{\sqrt{\lambda_{\max}(M_i)\lambda_{\max}(G)}}{\phi_i} V_i(\tilde{\boldsymbol{\zeta}}^i(0))^{\frac{1}{2}} \\ &\leq \sqrt{2}\frac{\lambda_{\max}(M_i)\lambda_{\max}(G)}{\phi_i}\tilde{\boldsymbol{\zeta}}^i(0)\end{aligned}$$

This completes the proof. □

At this point, the following result on the boundedness of the state estimation error of each agent i holds when considering a bounded input map $g(\cdot)$ as in Assumption 1-a).

Corollary 2. *Consider a multi-agent system with dynamics as in* (18). *Let each agent i run a distributed state observer as in* (33) *under Assumption 1-a), that is with known bounded input map $g(\cdot)$. Then, it is*

$$\|\tilde{\boldsymbol{x}}^i(t)\| \leq \|\tilde{\boldsymbol{x}}^i(0)\|, \quad \forall\, t \geq 0 \tag{103}$$

and there exists $T_x > 0$ such that

$$\|\tilde{\boldsymbol{x}}^i(t)\| = 0, \quad \forall\, t > T_x \tag{104}$$

with $T_x = \max_{i \in V}\{T_{x,i}\}$.

Proof. This result follows straightforwardly from the definition of vector $\boldsymbol{\zeta}^i$ in eq. (41) and from Theorem 4. $\square$

Remark 3. Theorem 4 demonstrates that the proposed local observer exhibits finite-time convergence under a boundedness condition of the estimate $K[\tilde{\boldsymbol{\varrho}}^i]$ of the input map. According to Assumption 1, two different scenarios arise, namely that either $g(\cdot)$ is bounded or its derivative $K[\dot{g}](\cdot)$ is bounded. In the first case, the input map $g(\cdot)$ can be considered as a pure disturbance, and the convergence of the state observer is ensured by the knowledge of an upper bound for $g(\cdot)$ from which an upper-bound of $K[\tilde{\boldsymbol{\varrho}}^i]$ follows. In the second case, we can design a finite-time observer for the input map $g(\cdot)$ and the convergence of the input observer is ensured by the the boundedness of the derivative $K[\dot{g}](\cdot)$. Furthermore, the convergence of the state observer follows from the the fact $K[\tilde{\boldsymbol{\varrho}}^i]$ goes to zero in finite-time.

C. k-hop Graph-Based Input Map Observer

In this section, we derive a finite-time distributed observer to estimate the agents' input map $g(\cdot)$ in eq. (18). Since the mathematical derivation follows from the reasoning proposed in Section V for the state observer, details concerning common aspects will be omitted for the sake of brevity.

Let us assume that agent i is willing to estimate the stacked vector of the input maps $g_j = g(u_j)$ of the agents j belonging to its k-hop augmented neighborhood $\bar{\mathcal{N}}_i^k$, i.e. $\boldsymbol{g}^i$ as defined in eq. (25). We reiterate that, in the case the input map $\boldsymbol{g}^i$ is not bounded, the convergence of the observer (33) requires the knowledge of the estimate $\hat{\boldsymbol{g}}^i$, as discussed in Remark 3.

Therefore, similarly to the observer proposed in Section V for $\boldsymbol{x}^i$, we design the following update law for stacked vector of input maps $\boldsymbol{g}^i$:

$$\begin{aligned}
\dot{\hat{\boldsymbol{g}}}^i &= \bar{\Pi}_i \operatorname{sign}\left(\boldsymbol{\eta}_i - \hat{\boldsymbol{g}}^i\right) \\
\boldsymbol{\eta}_i &= \bar{H}_i(\boldsymbol{g}^i - \hat{\boldsymbol{g}}^i) \\
&\quad + \sum_{j\in\mathcal{N}_i} \bar{P}_i\left(-\bar{P}_j^T \bar{P}_j \bar{P}_i^T \hat{\boldsymbol{g}}^i + \bar{P}_i^T \bar{P}_i \bar{P}_j^T \hat{\boldsymbol{g}}^j\right) + \hat{\boldsymbol{g}}^i
\end{aligned} \tag{105}$$

with $\bar{\Pi}_i = \Pi_i \otimes I_d$ and Π_i a diagonal tuning matrix such as:

$$\Pi_i = \operatorname{diag}\{\pi_{n_i^1}, \ldots, \pi_{n_i^{\eta_i}}\} \tag{106}$$

and $\pi_{n_i^l} \in \mathbb{R}^+$. Moreover, let us define the estimation error $\tilde{\boldsymbol{g}}^i = \boldsymbol{g}^i - \hat{\boldsymbol{g}}^i$. We are now ready to demonstrate the finite-time convergence of the k-hop local observer given in eq. (105).

Theorem 5. *Let us consider the observer error dynamics described in eq.* (105)*. According to Assumption 1-b), let us denote with d_2 the bound of the derivative $K[\dot{g}](\cdot)$ of the input map, i.e., $\|K[\dot{g}](\cdot)\| \le d_2$. Then $\tilde{\boldsymbol{\varrho}}^i$ as in eq.* (49) *reaches the origin in finite time $T_{g,i}$*

$$T_{g,i} \le \sqrt{2}\frac{\lambda_{\max}(M_i^k)}{\psi_i}\tilde{\boldsymbol{\varrho}}^i(0)$$

given that the gain π_i satisfies

$$\pi_i > \frac{\lambda_{\max}(M_i)}{\lambda_{\min}(M_i)}\sqrt{\eta_i}\, d_2 \tag{107}$$

Proof. In order to prove the finite-time convergence of the input map observer, we follow a similar reasoning as before. Therefore, in the following we sketch only the main aspects of the analysis. Briefly, as pointed out in Lemma 1 for the state estimation, also for the input map estimation it is possible to change point of view. That is, instead of proving that each agent i is able to estimate the input map of the agents belonging to its k-hop augmented neighborhood, namely $\boldsymbol{g}^i$, it can be proved that the local input map $g_i = g(u_i)$ of each agent i is estimated by the agents belonging to $\mathcal{V}_i$ as previously defined. Therefore, by considering the vector $\hat{\boldsymbol{\varrho}}^i$ containing the input map estimates made by the agents belonging to $\mathcal{V}_i$ as defined in eq. (48) and by following exactly the same steps as in the proof of Lemma 3, we obtain

$$\dot{\hat{\boldsymbol{\varrho}}}^i = \pi_i \operatorname{sign}\left((L_i + H_i) \otimes I_d)\tilde{\boldsymbol{\varrho}}^i\right) = \pi_i \operatorname{sign}\left((M_i \otimes I_d)\,\tilde{\boldsymbol{\varrho}}^i\right) \tag{108}$$

from which by recalling the definition of $\tilde{\boldsymbol{\varrho}}^i$ as in eq. (49) we have

$$\dot{\tilde{\boldsymbol{\varrho}}}^i = -\,\pi_i \operatorname{sign}\left((M_i \otimes I_d)\,\tilde{\boldsymbol{\varrho}}^i\right) + \dot{\boldsymbol{\varrho}}^i \tag{109}$$

Thus, a similar structure as in eq. (51) is obtained, where I_d appears instead of G.

Let us now consider the following Lyapunov candidate function

$$V_i(\tilde{\boldsymbol{\varrho}}^i) = \frac{1}{2}\tilde{\boldsymbol{\varrho}}^{iT}\,(M_i \otimes I_d)\;\tilde{\boldsymbol{\varrho}}^i \tag{110}$$

for which according to Theorem 2, the generalized derivative $\mathring{V}_i(\tilde{\boldsymbol{\varrho}}^i)$ is computed as

$$\mathring{V}_i(\tilde{\boldsymbol{\varrho}}^i) = \left(\nabla V_i(\tilde{\boldsymbol{\varrho}}^i)\right)^T K\left[\dot{\tilde{\boldsymbol{\varrho}}}^i\right] = \tilde{\boldsymbol{\varrho}}^{iT}\,(M_i \otimes I_d) K\left[\dot{\tilde{\boldsymbol{\varrho}}}^i\right] \tag{111}$$

At this point, by exploiting eq. (109) along with the sum property and the vector property of the Filippov calculus given in Theorem 3 we have

$$\begin{aligned}
\mathring{V}_i(\tilde{\boldsymbol{\varrho}}^i) &\subset \tilde{\boldsymbol{\varrho}}^{iT}\,(M_i \otimes I_d)\, K[-\pi_i \operatorname{sign}\left((M_i \otimes I_d)\,\tilde{\boldsymbol{\varrho}}^i\right)] \\
&\quad + \tilde{\boldsymbol{\varrho}}^{iT}\,(M_i \otimes I_d)\, K[\dot{\boldsymbol{\varrho}}^i] \\
&\subset \tilde{\boldsymbol{\varrho}}^{iT}\,(M_i \otimes I_d)\, K[-\pi_i \operatorname{sign}\left((M_i \otimes I_d)\,\tilde{\boldsymbol{\varrho}}^i\right)] \\
&\quad + \tilde{\boldsymbol{\varrho}}^{iT}\,(M_i \otimes I_d)\,(1_{\eta_i} \otimes K[\dot{g}_i])
\end{aligned} \tag{112}$$

where eq. (47) has been used for applying the vector property. Now, by performing similar manipulations as in the proof of Theorem 4, we obtain:

$$\begin{aligned}
\mathring{V}_i(\tilde{\boldsymbol{\varrho}}^i) &\le -\pi_i\, \|(M_i \otimes I_d)\tilde{\boldsymbol{\varrho}}^i\,\|_1 \\
&\quad + \|\tilde{\boldsymbol{\varrho}}^i\|\|\,(M_i \otimes I_d)\,\|\|(1_{\eta_i} \otimes K[\dot{g}_i])\| \\
&\le -\pi_i\, \|(M_i \otimes I_d)\tilde{\boldsymbol{\varrho}}^i\,\|_1 \\
&\quad + \|\tilde{\boldsymbol{\varrho}}^i\|\|\,(M_i \otimes I_d)\,\|\sqrt{\eta_i}\, d_2
\end{aligned} \tag{113}$$

where the next facts have been exploited to derive the last inequality $\|(1_{\eta_i} \otimes K[\dot{g}_i])\| \le \sqrt{\eta_i}\,\|K[\dot{g}_i]\|$ and $\|K[\dot{g}_i]\| \le d_2$. At this point, the following bound on the generalized derivative $\mathring{V}_i(\tilde{\boldsymbol{\varrho}}^i)$ holds

$$\mathring{V}_i(\tilde{\boldsymbol{\varrho}}^i) \le -\psi_i\,\|\tilde{\boldsymbol{\varrho}}^i\| \tag{114}$$

where ψ_i is defined as

$$\psi_i = [\pi_i\,\lambda_{\min}(M_i) - \|\ M_i \otimes I_d\ \|\ \sqrt{\eta_i}\, d_2] \tag{115}$$

Similarly to the proof of Theorem 4, we obtain the final result

$$\mathring{V}_i(\tilde{\boldsymbol{\varrho}}^i) \le -\frac{\psi_i}{\sqrt{\lambda_{\max}(M_i)}} V_i(\tilde{\boldsymbol{\varrho}}^i)^{\frac{1}{2}} \tag{116}$$

from which the finite-time convergence of the Lyapunov function to the origin holds if

$$\pi_i > \frac{\lambda_{\max}(M_i)}{\lambda_{\min}(M_i)} \sqrt{\eta_i}\, d_2 \tag{117}$$

with settling time

$$T_{g,i} = 2\frac{\sqrt{\lambda_{\max}(M_i^k)}}{\psi_i} V_i(\tilde{\boldsymbol{\varrho}}^i(0))^{\frac{1}{2}} \leq \sqrt{2}\frac{\lambda_{\max}(M_i^k)}{\psi_i} \tilde{\boldsymbol{\varrho}}^i(0)$$

□

Remark 4. Theorem 5 provides conditions for the convergence of the finite-time observer of the input map. It should be noticed that the proposed reasoning could be iterated in order to relax the boundedness condition on the input map and its first r derivatives, by assuming the existence of an upper bound on the $r+1$-th derivative and deriving an observer of higher order. In our opinion, this represents a good compromise between computational complexity and strictness of the assumptions, i.e., input map boundedness or its derivatives boundedness that can be taken into consideration in the network design.

At this point, the following result holds on the boundedness of the input estimation error for each agent i.

Corollary 3. *Consider a multi-agent system with dynamics as in* (18)*. Let each agent i run an input observer as in* (105) *under Assumption 1-b). Then,*

$$\|\boldsymbol{g}^i(t)\| \leq \|\boldsymbol{g}^i(0)\|, \quad \forall\, t \geq 0 \tag{118}$$

and there exists $T_g > 0$ such that

$$\|\boldsymbol{g}^i(t)\| = 0, \quad \forall\, t > T_g \tag{119}$$

with $T_g = \max_{i\in V}\{T_{g,i}\}$.

Proof. This result follows straightforwardly from the definition of vector $\tilde{\boldsymbol{\varrho}}^i(t)$ in eq. (49) and from Theorem 5. □

In addition, the following result holds for a multi-agent system running the state observer together with the input observer, by combining Corollary 2 and Corollary 3.

Lemma 5. *Consider a multi-agent system with dynamics as in* (18)*. Assume each agent i runs a distributed state observer as in* (33) *and input dependent observer as in* (105) *under Assumption 1-b). Then, there exist $T_g > 0$ and $\mathcal{X} > 0$ such that*

$$\|\tilde{\boldsymbol{x}}^i(t)\| \leq \mathcal{X}, \quad \forall\, t > T_g \tag{120}$$

with $T_g = \max_{i\in V}\{T_{g,i}\}$ as per Corollary 3 and $\mathcal{X}$ defined as

$$\mathcal{X} = \max_{i\in V}\left\{\sup_{0\leq\tau\leq T_g} \|\tilde{\boldsymbol{x}}^i(\tau)\|\right\}. \tag{121}$$

Furthermore, there exists $T_{gx} > 0$ such that

$$\|\tilde{\boldsymbol{x}}^i(t)\| = 0, \quad \forall\, t > T_{gx} \tag{122}$$

with $T_{gx} = T_g + T_x$ and T_x defined in Corollary 2.

Proof. To prove the Lemma, we notice that for generalized Lyapunov function derivative in Theorem 4 to be negative definite, the condition given in eq. (102) must be satisfied, see eqs. (98)–(101) for further details.

In this regard, from Corollary 3 we know that there exists time T_g such that this condition will be satisfied for any $t > T_g$ (and for any choice of $\theta_i > 0$). This also ensures that starting from T_g the state estimation error $\tilde{\boldsymbol{\zeta}}^i$ will be decreasing for any agent i, that is $\tilde{\boldsymbol{\zeta}}^i(t) \leq \tilde{\boldsymbol{\zeta}}^i(T_g)$ with $t \geq T_g$.

Let us now demonstrate that the Lyapunov function remains finite over this time interval. To this end, let us consider again the inequality given for the generalized Lyapunov derivative in eq. (98), that is

$$\begin{aligned} \mathring{V}_i(\tilde{\boldsymbol{\zeta}}^i) &\leq -\theta_i\, \lambda_{\min}(M_i)\, \lambda_{\min}(G)\, \|\, \tilde{\boldsymbol{\zeta}}^i \,\| \\ &\quad + \|\tilde{\boldsymbol{\zeta}}^i\| \|\, (M_i \otimes G)\, \| \|K[\tilde{\boldsymbol{\varrho}}^i]\| \\ &\leq -\phi_i\, \|\, \tilde{\boldsymbol{\zeta}}^i \,\| \end{aligned} \tag{123}$$

In particular, since the condition given in eq. (102) may not hold in the time time-interval $[0,\, T_g]$, it follows that ϕ_i as defined in (99) may be negative, that is $\phi_i < 0$.

Therefore, the Lyapunov function in (84) could increase accordingly in the time interval $[0,\, T_g]$, and so would do the state estimation error $\tilde{\boldsymbol{\zeta}}^i$. Nevertheless, since the generalized Lyapunov derivative $\mathring{V}_i(\tilde{\boldsymbol{\zeta}}^i)$ is upper-bound from above by a continuous positive quantity as given in (123), and the time interval is finite, it follows that the the Lyapunov function remains finite over this time-interval and thus also the estimation error $\tilde{\boldsymbol{\zeta}}^i$ remains finite. From this reasoning, it follows that an upper bound for the state estimation error $\tilde{\boldsymbol{\zeta}}^i$ of any agent i can be found as

$$\mathcal{X} = \max_{i\in V}\left\{\sup_{0\leq\tau\leq T_g} \|\tilde{\boldsymbol{x}}^i(\tau)\|\right\} \tag{124}$$

which is the largest value (in norm) that such an error may have achieved over this time interval $[0,\, T_g]$.

At this point, to prove the last part of the Corollary, it is sufficient to notice that for any time $t > T_g$, the conditions of Corollary 2 holds, and in particular

$$\|\tilde{\boldsymbol{x}}^i(t)\| \leq \|\tilde{\boldsymbol{x}}^i(T_g)\| \leq \mathcal{X}, \quad \forall\, t > T_g \tag{125}$$

and

$$\|\tilde{\boldsymbol{x}}^i(t)\| = 0, \quad \forall\, t > T_g + T_x \tag{126}$$

□

VI. Closed loop analysis

In many practical applications, the state estimated according to the observer described in the previous sections is used to implement a local feedback control law. Although, the stability of the coupled observer and controller scheme might not abstract from the particular agent dynamics and feedback control law, we investigate in this section some general conditions that might turn useful to make easier the design of the control law and to guarantee the overall stability of the multi-agent dynamics when closing the loop through the state observer.

Consider the following stacked multi-agent dynamics

$$\dot{\boldsymbol{x}} = \bar{f}(\boldsymbol{x}) + (I_n \otimes A)\, \boldsymbol{x} + \boldsymbol{g}(\boldsymbol{u}) \tag{127}$$

where each agent i has a non-linear dynamics of the form (18) and

$$\boldsymbol{g}(\boldsymbol{u}) = \begin{bmatrix} g(u_1)^{\mathrm{T}} & \dots & g(u_n)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$$

Let us consider a general nonlinear state-feedback $\boldsymbol{u} = \boldsymbol{q}(\boldsymbol{x})$ defined as

$$\boldsymbol{u} = \boldsymbol{q}(\boldsymbol{x}) = \begin{bmatrix} q_1(\boldsymbol{x}^1)^{\mathrm{T}} & \cdots & q_n(\boldsymbol{x}^n)^{\mathrm{T}} \end{bmatrix} \tag{128}$$

where each agent i needs to exploit k-hop information to compute its control input $u_i = q_i(\boldsymbol{x}^i)$ with $\boldsymbol{x}^i$ defined as in (19).

In particular, the following assumption is taken on the function q_i, with $i \in \{1, \ldots, n\}$.

Assumption 3. One of the following conditions must hold:

a) the function $q_i(\cdot)$ is bounded (with a known upper-bound);
b) the derivative $K[\dot{q}_i](\cdot)$ is bounded (with a known upper-bound).

Note that Assumption 3 is required when closing the loop with a state-feedback control of the form $u_i = q_i(\boldsymbol{x}^i)$ to ensure the satisfaction of Assumption 1. In particular, Assumption 3-a) suffices to ensure the satisfaction of Assumption 1-a); while Assumption 3-b) suffices to ensure the satisfaction of Assumption 1-b).

Remark 5. Note that, since $\boldsymbol{x}^i$ is not locally available, the following control is implemented instead

$$u_i = q_i(\hat{\boldsymbol{x}}^i) = q_i(\boldsymbol{x}^i - \tilde{\boldsymbol{x}}^i) \tag{129}$$

with $\hat{\boldsymbol{x}}^i$ the estimation of $\boldsymbol{x}^i$ made by agent i as defined in eq. (22). We point out that the structure of the control input may differ from agent to agent (i.e., $q_i(\cdot) \neq q_j(\cdot)$, $i \neq j$). Furthermore, even if all agents' control input share the same structure $q_i(\cdot) = q(\cdot), \forall\, i \in \{1, \ldots, n\}$, their implementation will change since their local k-hop neighborhood will be different (i.e., $\boldsymbol{x}_i \neq \boldsymbol{x}_j$, $i \neq j$).

At this point, let us define the following function $\Phi(\boldsymbol{x}, \tilde{\boldsymbol{x}})$

$$\Phi(\boldsymbol{x}, \tilde{\boldsymbol{x}}) = \boldsymbol{f}(\boldsymbol{x}) + (I_n \otimes A)\, \boldsymbol{x} + \boldsymbol{g}(\boldsymbol{q}(\boldsymbol{x} - \tilde{\boldsymbol{x}})) \tag{130}$$

with $\tilde{\boldsymbol{x}}$ the disturbance (input) of the system and $\Phi(\boldsymbol{x}, 0_{nd})$ the nominal (unforced) dynamics represented by (127) and (128), i.e.,

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + (I_n \otimes A)\, \boldsymbol{x} + \boldsymbol{g}(\boldsymbol{q}(\boldsymbol{x})) = \Phi(\boldsymbol{x}, 0_{nd}) \tag{131}$$

The following assumption is now made.

Assumption 4. The nonlinear state-feedback $\boldsymbol{u} = \boldsymbol{q}(\boldsymbol{x})$ in (128) ensures the convergence of the multi-agent system towards an equilibrium representing a given team objective.

We are now ready to state the main theorem of this section.

Theorem 6. *Consider a multi-agent system with dynamics as in* (127) *Let each agent i runs a distributed state observer as in* (33) *and input observer as in* (105) *and implements the local control input as in* (129) *under Assumptions 1-b) and 3-b). Then, if the dynamics $\Phi(\boldsymbol{x}, \tilde{\boldsymbol{x}})$ is set Input-to-State Stable and Assumption 4 holds, the multi-agent system reaches an equilibrium representing the team objective.*

Proof. In order to prove the theorem, we resort on the well-known results of the set Input-to-State Stability (set-ISS) [**?**] In particular, from the set-ISS assumption of the dynamics $\Phi(\boldsymbol{x}, \tilde{\boldsymbol{x}})$, it follows that there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that for any initial state $\boldsymbol{x}(0)$ and any input $\tilde{\boldsymbol{x}}(t)$, the solution $\boldsymbol{x}(t)$ exists for all $t \geq 0$ and satisfies

$$\|\boldsymbol{x}(t)\|_{\mathcal{A}} \leq \beta\left(\|\boldsymbol{x}(0)\|_{\mathcal{A}}, t\right) + \gamma\left(\sup_{0 \leq \tau \leq t} \|\tilde{\boldsymbol{x}}(\tau)\|\right) \tag{132}$$

where $\|\boldsymbol{x}\|_{\mathcal{A}} = \operatorname{dist}(\boldsymbol{x}, \mathcal{A}) = \inf\limits_{\boldsymbol{a} \in \mathcal{A}} \{\|\boldsymbol{x} - \boldsymbol{a}\|\}$ denotes the point to set distance.

At this point, we notice that Lemma 5 holds under Assumptions 1-b) and 3-b). Therefore, we know that there exists an upper bound $\mathcal{X} \geq 0$, such that for all agents $\|\tilde{\boldsymbol{x}}^i(t)\| \leq \mathcal{X}$, which in turn implies that $\|\tilde{\boldsymbol{x}}(\tau)\| < \sqrt{n}\, \mathcal{X}$, $\forall\, t$. In addition, from Lemma 5 we also know that there exists $T_{gx} > 0$ such that $\|\tilde{\boldsymbol{x}}^i(t)\| = 0, \forall\, t > T_{gx}$.

Therefore, by considering $\boldsymbol{x}(T_{gx})$, we have that for any $t > T_{gx}$ the multi-agent system evolves from this new initial condition $\boldsymbol{x}(T_{gx})$ according to the nominal dynamics $\Phi(\boldsymbol{x}, 0)$:

$$\begin{aligned} \|\boldsymbol{x}(t)\|_{\mathcal{A}} &\leq \beta\left(\|\boldsymbol{x}(T_{gx})\|_{\mathcal{A}}, t - T_{gx}\right) + \gamma\left(\sup_{T_{gx} \leq \tau \leq t} \|\tilde{\boldsymbol{x}}(\tau)\|\right) \\ &\leq \beta\left(\|\boldsymbol{x}(T_{gx})\|_{\mathcal{A}}, t - T_{gx}\right) \end{aligned} \tag{133}$$

thus an equilibrium is achieved by construction and the result follows.

□

VII. SIMULATION CASE STUDIES

Let us consider a multi-agent system composed of $n = 100$ agents the interactions of which are described by an undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ representing a regular lattice network. Moreover, let us assume that it is $k = 2$ for the state observer (2-hop observer).

Let us assume that, each agent in eq. (18) has the following first order dynamical nonlinear model

$$\dot{x}_i = -\tanh(x_i^3) + g(u_i) \tag{134}$$

where $x_i, u_i \in \mathbb{R}$ and input mapping $g(\cdot)$ will be specified in the following.

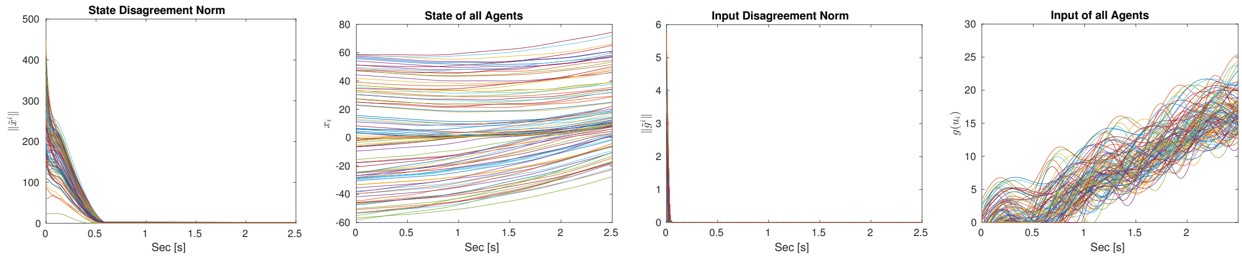
Two case studies are considered: i) the first one shows the fundamental properties of the observer with unbounded exogenous inputs, and ii) the second one demonstrates that we can safely close the loop on the estimated state to achieve a stable collective behavior. Observer parameters have been chosen to satisfy the conditions given in Theorems 4 and 5. In particular, gains in (36) have been chosen for all simulations as $\omega_{\eta_i^j} = 5$, $\theta_{\eta_i^j} = 130, \forall\, i = 1, 2, \cdots, n$ and $j \in \bar{\mathcal{N}}_i^k$, while it is $G = 3$ in (37) and $\pi_{\eta_i^j} = 120$ in (106) (in the case the input observer is running).

A. First case study: Unbounded control input

Figure 2 describes the simulation results for the second scenario, where each agent is driven by the exogenous input given by

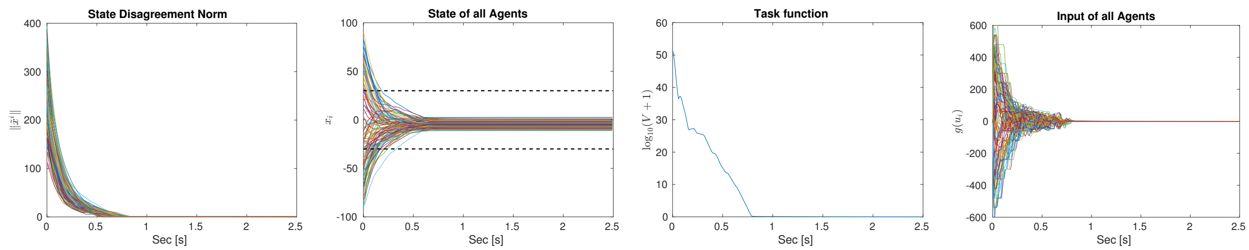
$$u_i(t) = s_i\, \sin(2\, \pi f_i\, t) + r_i\, t + u_{i0} \tag{135}$$

where s_i, f_i, r_i and u_{i0} are scalar parameter different for each agent and randomly generated. In this simulation case study, function $g_i(\cdot)$ in (134) is a classical dead zone non linearity with dead zone $[-5, 5]$ and unitary slope outside this interval. It is worth highlighting that functions $g(u_i)$ $(\forall\, i)$ meet Assumption 1-b) with $d_2 = \max_i\{2\pi f_i s_i + r_i\}$ in Theorem 5. Figures 2a and 2b describe the norm of the estimation error $\|\tilde{\boldsymbol{x}}^i\|$ and the states x_i, respectively; the first plot clearly shows the finite time converge of the state estimation error. Similarly, Figs. 2c and 2d describe the norm of the input estimation error $\|\tilde{\boldsymbol{g}}^i\|$ and the input g_i for all agents in the network. Each agent is driven by an unbounded exogenous input; then, to ensure the boundedness condition on the input estimate, required by



(a) **(b)** **(c)** **(d)**

Fig. 2: First Case Study. Simulation results for the first scenario where each agent is running the state and the input observers with exogenous unbounded input. In particular, Figs. 2a and 2b depict the observer error $\|\tilde{\boldsymbol{x}}^i\|$ and agent state x_i evolution, respectively. Fig. 2c shows the evolution of $\|\tilde{\boldsymbol{g}}^i\|$ and Fig. 2d shows input $g(u_i)$ for each agent of the network.



(a) **(b)** **(c)** **(d)**

Fig. 3: Second Case Study. Simulation results for the second scenario where each agent is running the state observer and the network exhibits a cooperative behavior. In particular, Figs. 3a and 3b depict the observer error $\|\tilde{\boldsymbol{x}}^i\|$ and agent state x_i evolution, respectively. Fig. 3c shows the evolution of $\log_{10}(V+1)$ with V defined in (139), and Fig. 3d shows input $g(u_i)$ for each agent of the network.

Theorem 4 for the finite-time convergence of the local state observer, we are required to let each agent run the input observer. In this regard, it can be notice that according to Theorem 5, the boundedness of the first-order derivative of the input suffices to guarantee the finite-time convergence of the local input observes, which in turn suffices to guarantee the finite-time convergence of the state observer.

B. Second case study: Cooperative behavior

In this case study, a stable cooperative behavior for large scale network is designed. In particular, the scope of the control input is described in the following.

Problem 3. Consider a multi-agent system composed of n agents. Our objective is to define a control law such that the following holds true:

- The multi-agent system must reach an equilibrium where the agents' state is within a predefined range, let's say $[x_{\min}, x_{\max}]$;
- Each agent i must reach a final state x_i which does not differ from the state of any of its k-hop neighbors $x_j \in \bar{\mathcal{N}}_i^2$, more than a given threshold ε, that is $|x_i - x_j| \leq \varepsilon, \forall\, i, j$.

Remark 6. Problem 3 differs from classical distributed consensus-based problem setting which can be solved by 1-hop interaction rules as it requires k-hop neighborhood information (in this case $k=2$) usually not available to the agents.

To the aim, let us consider the error $\tilde{x}_{i,j} = x_i - x_j$ and a smooth monotonically increasing function $\delta_{i,j} = \delta(\tilde{x}_{i,j})$, $\forall\, i, j \leq n$, with the following properties:

Property 1. *$\delta_{i,j}$ is anti-symmetric, i.e., $\delta_{i,j} = -\delta_{j,i}$;*

Property 2. *$\delta_{i,j} = 0$ ($\delta'_{i,j} = \partial\delta_{i,j}/\partial\tilde{x}_{i,j} = 0$) for $|\tilde{x}_{i,j}| = |x_i - x_j| \leq \varepsilon$ and $\delta_{i,j} \neq 0$ ($\delta'_{i,j} > 0$) otherwise.*

Moreover, let us introduce a leader agent with constant state equal to $x_{n+1} = (x_{max} + x_{min})/2$ and function $\delta_{i,n+1}$ $\forall\, i$ defined as $\delta_{i,j}$ but with $\varepsilon = (x_{max} - x_{min})/2$.
An example of such function is

$$\delta_{i,j} = \begin{cases} \operatorname{sign}(\tilde{x}_{i,j})\cosh(|\tilde{x}_{i,j}| - \varepsilon), & |\tilde{x}_{i,j}| > \varepsilon \\ 0, & |\tilde{x}_{i,j}| \leq \varepsilon \end{cases} \tag{136}$$

We now provide a control law capable of solving Problem 3 under the assumption that each agent i knows the state of its 2-hop neighborhood. Then, we will show that the proposed control law can still be used when exploiting the proposed finite-time distributed observer.

Theorem 7. *Let us consider a system of n agents with dynamics as in* (134)*, $g(u_i) = u_i$ and whose communication graph is undirected and connected. In the case the control input is selected as:*

$$u_i = \tanh(x_i^3) - \gamma \tanh\left(\sum_{j \in \overline{\mathcal{N}}_i^2} \delta'_{i,j}\delta_{i,j}\right) \tag{137}$$

with γ a scalar positive gain, then Problem 3 is solved.

Remark 7. Functions $g(u_i)$ ($\forall\, i$) meet Assumption 1-a), and the control input (137) operates a feedback linearization to cancel out the bounded non linear term $-\tanh(x_i^3)$. We point out that the choice of using a simple control approach was made on purpose, since the scope of this case study is not to solve a particular control problem but to present an application of the devised framework. Thus, we preferred to keep the control part as simple as possible for the sake of clarity.

Proof. To prove the theorem we notice that Problem 3 is solved if $\delta_{i,j} = 0$, $\forall i \in \{1, \ldots, n\}$ and $j \in \overline{\mathcal{N}}_i^2 \cup \{n+1\}$. In this regard, by plugging (137) into the dynamics in (134), it is

$$\dot{x}_i = -\gamma \tanh\left(\sum_{j \in \overline{\mathcal{N}}_i^2} \delta'_{i,j}\delta_{i,j}\right) \tag{138}$$

Let us consider the following Lyapunov function

$$V = \frac{1}{2}\sum_i \sum_{j\in\overline{\mathcal{N}}_i^2} \delta_{i,j}^2 \tag{139}$$

that is zero when all $\delta_{i,j} = 0$, $\forall\, i \in \{1, 2, \ldots, N\}$ and $\forall\, j \in \overline{\mathcal{N}}_i^2$. In particular, the time derivative of (139) takes the following form

$$\begin{aligned}\dot{V} &= -\gamma \sum_i \sum_{j\in\overline{\mathcal{N}}_i^2} \delta_{i,j}\delta'_{i,j}(\dot{x}_i - \dot{x}_j) \\ &= -\gamma \sum_i \sum_{j\in\overline{\mathcal{N}}_i^2} \delta_{i,j}\delta'_{i,j}\left[\tanh\left(\sum_{k\in\overline{\mathcal{N}}_i^2}\delta'_{i,k}\delta_{i,k}\right)\right. \\ &\qquad\qquad \left. - \tanh\left(\sum_{k\in\overline{\mathcal{N}}_j^2}\delta'_{j,k}\delta_{j,k}\right)\right]\end{aligned} \tag{140}$$

By performing some manipulation the previous equation can be re-written as

$$\dot{V} = -2\gamma \sum_i \left[\left(\sum_{j\in\overline{\mathcal{N}}_i^2}\delta_{i,j}\delta'_{i,j}\right)\tanh\left(\sum_{j\in\overline{\mathcal{N}}_i^2}\delta'_{i,j}\delta_{i,j}\right)\right] \tag{141}$$

where the negative-semi definiteness simply follows from the oddness of the $\tanh$ function. At this point, by the Lasalle's invariance principle, V is bounded with positive invariant set $D = \{\boldsymbol{x} \in \mathbb{R}^{dn} \,|\, \dot{V} = 0\}$. By looking at the structure of $\dot{V}$, we notice that $\dot{V} = 0$ implies

$$\sum_{j\in\overline{\mathcal{N}}_i^2} \delta'_{i,j}\delta_{i,j} = 0,\ \forall\, i \in \{1,\, \ldots,\, n\} \tag{142}$$

Moreover, from Properties 1 and 2 it follows that $\delta_{i,j}$ is an odd function, $\delta'_{i,j}$ is non-negative and symmetric, and since the communication graph is connected, from eq. (142) it follows that $\delta_{i,j} = 0$, $\forall\,(i,j) \in \overline{\mathcal{N}}_i^2$ with $i \in \{1,\, \ldots,\, n\}$. Thus proving the statement. □

Remark 8. Proof of Theorem 7 has been provided under the assumption of the availability of the k-hop neighborhood state as stated in Section VI. In order to demonstrate the closed-loop stability when using the observed state $\hat{\boldsymbol{x}}^i$ (rather than the actual one $\boldsymbol{x}^i$) in the control law given in eq. (137), the set-ISS property with respect to the set $\mathcal{A} = [x_{\min},\, x_{\max}]$ needs to be shown as required by Theorem 6. Concerning this set-ISS property, it naturally holds for the system at hand since the adoption of the control law in eq. (137) leads to the following dynamics

$$\dot{\boldsymbol{x}} = -\begin{bmatrix}\tanh\left(\sum_{j\in\overline{\mathcal{N}}_1^2}\delta'_{1,j}\delta_{1,j}\right) \\ \vdots \\ \tanh\left(\sum_{j\in\overline{\mathcal{N}}_n^2}\delta'_{n,j}\delta_{n,j}\right)\end{bmatrix} = \boldsymbol{\phi}(\boldsymbol{x}, 0_n) \tag{143}$$

which is set-ISS with respect to the set $\mathcal{A}$ since V in eq. (139) is a set-ISS Lyapunov function with respect to $\mathcal{A}$ ([**?**], Theorem 2)

VIII. Conclusion

A finite-time distributed observer to let each agent estimate the k-hop neighborhood state by means of only local interaction has been described. Then, a closed loop analysis to investigate under which conditions the stability of the multi-agent dynamics can be achieved when closing the loop through the state observer has been provided. The proposed framework represents an effective solution to implement distributed control strategies in large-scale networked systems where due to the locality of the phenomena each agent is required to retrieve information concerning only a portion of the network to implement its own control strategy. Future work will be focused on extending the proposed framework to the case of switching topologies. Moreover, networking phenomena, such as packet drops and/or packets delays will be modelled and taken into consideration in the convergence proof.

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