Generalized Topology Control for Nonholonomic Teams with Discontinuous Interactions

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***Abstract*—In this paper, we consider the problem of *general* topology control in multi-robot systems with nonholonomic kinematics. Our contribution is twofold: we first demonstrate the correctness of topology control under the assumption that the network topology can switch *arbitrarily* and that potential-based mobility is *discontinuous* with respect to topology changes; we then demonstrate that a multi-robot team under the above listed conditions continues to achieve topology control when actuator saturation is applied and in the presence of arbitrary discontinuous (and possibly non-pairwise) exogenous objectives. Simulation results are given to corroborate our theoretical findings.**

***Index Terms*—Multi-Robot Systems; Topology Control; Nonsmooth Analysis.**

I. Introduction

COLLECTIVE multi-robot (-agent) motion has received significant attention in recent years, due to the scalability and robustness arising from the local nature of such systems. A wide array of multi-robot research trends have emerged, including area coverage [2], target tracking [3], and topology control [4]. Several important assumptions and shortcomings of multi-robot system control have been solved over the past decade, ranging from requirements for team interaction graphs (e.g., rigidity [5]), to assumptions on how proximity limitations impact agent motion (e.g., switching interaction graphs [6]).

The first goal of this paper is to illustrate a *general* topology control scheme for teams of *nonholonomic* robots. By *general* we mean possessing the capability to achieve various constraints on the interaction graph, including connectivity, rigidity, neighborhood cardinality, etc. Various works have investigated multi-robot topology control, including local connectivity control [7], [8], global connectivity control based on both discrete and algebraic methods [4], [9], and generic rigidity control [10], [11]. Likewise, nonholonomic kinematics have been studied in many multi-robot contexts, such as [8], [12], [13]. While there have been works to combine *local* topology control with nonholonomic constraints, as in the connectivity control of [8], general topology control under nonholonomic constraints remains an open problem. Our first contribution will thus demonstrate the nonholonomic extension of the mobility-based link control framework [14], allowing for the control of arbitrary topological constraints through team motion.

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The second goal of this work is to allow the network topology of the system to switch *arbitrarily*, with potential-based mobility being *discontinuous* with respect to topology changes. Such discontinuities are a *reality* of spatially interacting systems and can arise from non-ideal communication, proximity-limited inter-robot sensing, heterogenous robot mobility, unexpected failures, etc. In treating discontinuities in collaborative systems, works such as [15] consider finite switching, allowing classical analysis for differentiable dynamical systems to identify an equilibrium set of robot motions. Alternatively, for example in [16], works allow for infinite switching however under the assumption that consecutive network switches occur with a bounded *dwell-time*. It is also very common to assume the smoothness of potential fields with respect to proximity limitations, as in [8], [14]. Finally, as in the well-known flocking work [6], although the network is free of switching restrictions, it remains driven by potential field interactions that are *necessarily* continuously differentiable with respect to topology changes. We also mention the work [17] which is thematically related, where nonsmooth techniques are applied to ensure collision avoidance, yielding a new family of *centralized* navigation functions.

The final goal is to demonstrate that a multi-robot team under the above listed conditions continues to achieve topology control when actuator saturation is applied and in the presence of arbitrary discontinuous (and possibly non-pairwise) exogenous inputs. Our analysis of exogenous inputs also suffices to conclude robustness to measurement noise and actuator disturbance. Saturation and exogenous disturbance have been addressed in previous work, as in [18], [19], most commonly through Input-To-State-Stability (ISS) type results. In this work, we make a step forward by addressing these aspects together in a discontinuous context under general topology control.

II. Preliminaries

A. Agent and Network Modeling

Consider a system of n mobile nonholonomic ground robots operating over $\mathbb{R}^2$. The state of each robot i is described by its pose $p_i(t) = [x_i(t),\, y_i(t),\, \theta_i(t)]^T \in \mathbb{R}^2 \times (-\pi,\, \pi]$, where $q_i(t) = [x_i(t),\, y_i(t)]^T \in \mathbb{R}^2$ and $\theta_i(t) \in (-\pi,\, \pi]$ denote the position and orientation of the robot $i \in \mathcal{I} = \{1, \ldots, n\}$ at time $t \in \mathbb{R}_+$, respectively. Each robot i moves according to the following unicycle kinematics:

$$\begin{aligned} \dot{x}_i(t) &= v_i(t)\cos\theta_i(t) \\ \dot{y}_i(t) &= v_i(t)\sin\theta_i(t) \\ \dot{\theta}_i(t) &= \omega_i(t) \end{aligned} \tag{1}$$

where $u_i(t) = [v_i(t)\,\omega_i(t)]^T \in \mathbb{R}^2$ is the velocity control input for the robot i, with $v_i(t)$ and $\omega_i(t)$ the linear velocity and the angular velocity at time $t \in \mathbb{R}_+$, respectively . The aggregate state of the system is then given by $\mathbf{p} = [p_1^T, \ldots, p_n^T]^T$, the stacked vector of robot poses. Furthermore, we denote with $\mathbf{q} = [q_1^T, \ldots, q_n^T]^T$ the stacked vector of robots positions.

Assume the robots can interact in a proximity-limited way, inducing a topology of a time varying nature. Specifically, letting $d_{ij} \triangleq \|q_{ij}\| \triangleq \|q_i - q_j\|$ denote the Euclidean distance between robots i and j, and (i, j) a link between interacting robots. The spatial neighborhood of each robot is partitioned by defining concentric radii $\rho_2 > \rho_1 > \rho_0$. The radii introduce a *hysteresis* in interaction by assuming that links (i, j) are established only after $d_{ij} \leq \rho_1$, with link loss then occurring when $d_{ij} > \rho_2$, generating the annulus of $\rho_2 - \rho_1$ where *decisions* on link additions and deletions are made.

The assumed spatial interaction model is formalized by the *undirected dynamic graph*, $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$, with vertices (nodes) $\mathcal{V} = \{1, \ldots, n\}$ indexed by $\mathcal{I}$ (the robots), and edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ such that $(i, j) \in \mathcal{E} \Leftrightarrow (\|q_{ij}\| \leq \rho_2) \wedge (\sigma_{ij}(t) = 1)$, with switching signals $\sigma_{ij}(t) = 0$ if $(i, j) \notin \mathcal{E} \;\wedge\; \|q_{ij}\| > \rho_1$, or $\sigma_{ij}(t) = 1$ otherwise, with $\wedge$ the logical operator "AND". Finally, we assume $(i, i) \notin \mathcal{E}$ (no self-loops) and $(i, j) \in \mathcal{E} \Leftrightarrow (j, i) \in \mathcal{E}$ (symmetry) hold for all $i, j \in \mathcal{V}$. Nodes with $(i, j) \in \mathcal{E}$ are called *neighbors* and the neighbor set for an robot i is denoted $\mathcal{N}_i(t) = \{j \in \mathcal{V} \,|\, (i, j) \in \mathcal{E}(t)\}$.

B. Nonsmooth Analysis

We now briefly review the necessary machinery from nonsmooth analysis to prove our results. For a comprehensive overview the reader is referred to [20]–[22] and references therein. Consider the following dynamical system:

$$\dot{x} = f(x) \tag{2}$$

where $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is a *measurable* and *essentially locally bounded* function. Note that, if the map $f(\cdot)$ is continuous, then every solution $x(t)$ is continuously differentiable. On the contrary, for a map $f(\cdot)$ that is only measurable and essentially locally bounded the classical definition of a solution of differential equations no longer holds. Instead, the following notion of *Filippov* solution can be applied.

Definition 2.1 (Filippov Solution): A function $x(\cdot)$ is called a solution of (2) on $[t_0, t_1]$ if $x(\cdot)$ is absolutely continuous on $[t_0, t_1]$ and for almost all $t \in [t_0, t_1]$ it holds $\dot{x} \in K[f](x)$ with $K[f] : \mathbb{R}^n \to \mathfrak{B}(\mathbb{R}^n)$ the Filippov set-valued map defined as:

$$K[f](x) \equiv \bigcap_{\delta > 0} \bigcap_{\mu(Z) = 0} \overline{\text{co}}\,\{f(B(x, \delta) \setminus Z)\} \tag{3}$$

where $\bigcap_{\mu(Z)=0}$ denotes the intersection over all sets Z of Lebesgue measure zero, $\overline{\text{co}}$ denotes the convex closure and $\mathfrak{B}(\mathbb{R}^n)$ denotes the collection of subsets of $\mathbb{R}^n$, and $B(x, \delta)$ denotes an open ball centered at x of radius δ.

Next, in order to consider a locally Lipschitz function as a candidate Lyapunov function, *Clarke's generalized gradient* is defined as follows [23]:

Definition 2.2 (Clarke's generalized gradient): Let $V : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz continuous. Define the generalized gradient of V at x by

$$\partial V(x) = \text{co}\left\{\lim_{i \to \infty} \nabla V(x_i) \,|\, x_i \to x,\, x_i \notin \Omega_V \cup Z\right\} \tag{4}$$

where the set Ω_V is the set of Lebesgue measure zero where the gradient of V does not exist, Z is an arbitrary set of zero measure (which can simplify computation), and $\text{co}\{\cdot\}$ denotes the convex hull.

At this point, we outline a chain rule for differentiating regular functions along the Filippov solution trajectories.

Theorem 2.1 (Chain Rule): Let $x(t)$ be a Filippov solution to $\dot{x} = f(x)$ with $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ measurable and essentially locally bounded, and $V : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz and in addition regular function. Then $V(x(t))$ is absolutely continuous and the time derivative $(d/dt)V(x(t))$ exists almost everywhere and $\frac{d}{dt}V(x(t)) \in^{\text{a.e.}} \dot{\tilde{V}}(x(t))$ where the set-valued map $\dot{\tilde{V}}(x(t))$ is the *generalized time derivative* defined as:

$$\dot{\tilde{V}}(x(t)) = \bigcap_{\xi \in \partial V(x)} \xi^T K[f](x). \tag{5}$$

Computing the $K[f](x)$ in our case involves maps $f(\cdot)$ that are expressed as sums, products, or compositions of other functions. We now review a calculus originally developed in [24] (and outlined in an extended way in [22]), which will prove instrumental for the technical developments of this work.

Definition 2.3 (Calculus): The map $K[f] : \mathbb{R}^n \to \mathfrak{B}(\mathbb{R}^n)$ has the following properties.

1) Assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ is *locally bounded*. Then $\exists\, Z_f \subset \mathbb{R}^n$, $\mu(Z_f) = 0$, such that $\forall\, Z \subset \mathbb{R}^n, \mu(Z) = 0$,
$$K[f](x) = \overline{\text{co}}\left\{\lim_{i \to \infty} f(x_i) \,|\, x_i \to x,\, x_i \notin Z_f \cup Z\right\} \tag{6}$$
2) Assume that $f,\, g : \mathbb{R}^n \to \mathbb{R}^n$ are *locally bounded*. Then
$$K[f + g](x) \subseteq K[f](x) + K[g](x) \tag{7}$$
3) Assume that $f_i : \mathbb{R}^n \to \mathbb{R}^n,\ i = 1, \ldots, n$ are *locally bounded*. Then
$$K\left[\times_{i=1}^n f_i\right](x) \subseteq \times_{i=1}^n K\left[f_i\right](x) \tag{8}$$
where the cartesian product notation and column vector notation are used interchangeably, that is $x_1 \times x_2 \times x_3 = [x_1^T, x_2^T, x_3^T]^T$ with $x_i \in \mathbb{R}^n$.
4) Assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ is *locally bounded* and $g : \mathbb{R}^n \to \mathbb{R}^n$ is *continuously differentiable*. Then
$$K[f \circ g](x) = K[f](g(x)). \tag{9}$$
where $\circ$ represents the function composition operator.
5) Assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ is *locally bounded* and $g : \mathbb{R}^n \to \mathbb{R}^{n \times d}$ (i.e. matrix valued) is *continuous*. Then
$$K[g \cdot f](x) = g(x)K[f](x) \tag{10}$$
where $g \cdot f(x) = g(x)f(x) \in \mathbb{R}^n$.
6) Let $V : R^n \to R$ be *locally Lipschitz continuous*; then
$$K[\nabla V](x) = \partial V(x) \tag{11}$$
7) Assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ is *continuous*. Then
$$K[f](x) = \{f(x)\} \tag{12}$$

III. Topology Control with Discontinuities

A. An Event-Driven Decision-Making Framework

Let us first give a very brief description of the coordinated topology control developed in [14], to which the reader is referred to for a detailed description. Assume we have a topological property over $\mathcal{G}$, denoted by $\mathbb{P}(\mathcal{G})$, which we would like to ensure holds over the system trajectories, where by topological property we mean some desirable structure of the graph describing the agent interactions, e.g., graph rigidity. Furthermore, consider a more granular partitioning of inter-agent neighborhoods defined by $d_{ij} \in [\rho_2^-, \rho_2]$, the outer decision region where networked computation decides if $\mathcal{E} \cup \{(i,j)\}$ satisfies topological constraint $\mathbb{P}$; and $d_{ij} \in [\rho_1, \rho_1^+]$, the inner decision region where networked computation decides if $\mathcal{E} \setminus \{(i,j)\}$ satisfies topological constraint $\mathbb{P}$, with $0 < \rho_0 < \rho_1 < \rho_1^+ < \rho_2^- < \rho_2$. When $d_{ij} = \rho_2$ a new interaction is possible to occur and an event is triggered for agents i and j to evaluate whether a new link (i,j) would be acceptable given the topological constraint $\mathbb{P}(\mathcal{G})$. Likewise, when $d_{ij} = \rho_1$ and $(i,j) \in \mathcal{E}$ it is possible to lose an existing interaction and an event is triggered for agents i and j to evaluate whether losing link (i,j) would be acceptable given the topological constraint $\mathbb{P}(\mathcal{G})$. Formally, *predicates* over link (i,j), P_{ij}^a and P_{ij}^d, are evaluated by applying cooperative computation in the network to determine safe link additions and deletions according to the constraint $\mathbb{P}(\mathcal{G})$. We remind the reader that a predicate is a mapping from a statement to a Boolean outcome. It is important to note that the design of the above radii must consider the physical characteristics of the robotic systems and their computational capabilities. As computation must terminate within the radii, system motion and computation *must* be compatible.

We only require here the notion of a *decision set*, $\mathcal{D}_i^a$ for link addition and $\mathcal{D}_i^d$ for link deletion, which contain for an agent i all neighbors $j \in \mathcal{N}_i$ for which repulsive or attractive control must be applied for edge maintenance. Furthermore, there are two high-level ideas from [14] which will guarantee topological property maintenance. First, the members of decision sets $\mathcal{D}_i^a$ and $\mathcal{D}_i^d$ must be rendered *invariant* through robot mobility. That is, the topology control mechanism must preserve membership of all agents in these decision sets for the sake of level set invariance of the Lyapunov function. Under this invariance condition, the predicates must then be chosen to represent *worst-cases* with respect to a topological constraint:

Definition 3.1 (Worst-case graph, [14]): Assume we have a topological constraint $\mathbb{P}$ and a known space of reachable topologies $\mathcal{G}_{\mathbb{P}}(t)$ in which the network graph must lie, $\mathcal{G}(t) \in \mathcal{G}_{\mathbb{P}}(t)$, for all $t \geq 0$. A *worst-case* graph relative to constraint $\mathbb{P}$ is a graph $\mathcal{G}_w \in \mathcal{G}_{\mathbb{P}}$ such that $\mathbb{P}(\mathcal{G}_w) = 1 \Rightarrow \mathbb{P}(\mathcal{G}) = 1, \forall \mathcal{G} \in \mathcal{G}_{\mathbb{P}}$.

Now, the predicates must be constructed along the following restrictions:

Assumption 1 (Preemption, [14]): If in $\mathcal{G}_{\mathbb{P}}$ we can identify a worst-case graph, we must choose P_{ij}^a, P_{ij}^d (and by extension, the members of previously defined decision sets) such that $\mathbb{P}(\mathcal{G}_w) = 1$, which implies that $\mathbb{P}(\mathcal{G}) = 1, \forall \mathcal{G} \in \mathcal{G}_{\mathbb{P}}$, and network constraints are satisfied.

Let us now consider the following control law for maintaining a desired topological property with unicycle kinematics (guideline for this control law comes originally from [8]):

$$\begin{aligned} v_i &= -\big(f_{x_i}\cos\theta_i + f_{y_i}\sin\theta_i\big) \\ \omega_i &= -(\theta_i - \arctan2(f_{y_i}, f_{x_i})) \end{aligned} \tag{13}$$

where f_{x_i} and f_{y_i} are defined as:

$$\begin{aligned} f_{x_i} &= \sum_{j\in\mathcal{N}_i} \nabla_{x_i}\psi_{ij}^o + \sum_{j\in\mathcal{D}_i^d} \nabla_{x_i}\psi_{ij}^d + \sum_{j\in\mathcal{D}_i^a} \nabla_{x_i}\psi_{ij}^a \\ f_{y_i} &= \sum_{j\in\mathcal{N}_i} \nabla_{y_i}\psi_{ij}^o + \sum_{j\in\mathcal{D}_i^d} \nabla_{y_i}\psi_{ij}^d + \sum_{j\in\mathcal{D}_i^a} \nabla_{y_i}\psi_{ij}^a \end{aligned} \tag{14}$$

and $\psi_{ij}^o, \psi_{ij}^a, \psi_{ij}^d$ are potential fields for a general cooperative objective (e.g. swarming or coverage), link addition (attraction), and link deletion (repulsion), respectively. Further, the potentials are assumed to have the following properties.

Assumption 2 (Potential Field Properties): The potential fields $\psi_{ij}^h : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$, with $h \in \{o, d, a\}$ are assumed to be locally Lipschitz continuous and regular functions with properties:

$$\nabla_{q_i}\psi_{ij}^h = \nabla_{q_i}\psi_{ji}^h \quad \text{and} \quad \nabla_{q_i}\psi_{ij}^h = -\nabla_{q_j}\psi_{ij}^h \tag{15}$$

For the purposes of edge control, we only assume that $\psi_{ij}^a \to \infty$ as $d_{ij} \to \rho_1^+$ and $\psi_{ij}^d \to \infty$ as $d_{ij} \to \rho_2^-$. For collision avoidance, one can simply ensure that $\psi_{ij}^o \to \infty$ as $d_{ij} \to 0$.

B. Discontinuous Robot-to-Robot Interaction

The most natural points in which a system may experience discontinuity is when a discrete transition occurs and some reactive control must be applied. For our problem, such points occur when new agents are detected at radius ρ_2, current neighbors begin to move away at radius ρ_1, and when neighbors enter the collision region ρ_0. Thus, different than previous work, we assume that control functions $\psi_{ij}^o, \psi_{ij}^a, \psi_{ij}^d$ are *nonsmooth* at radii ρ_2, ρ_1, ρ_0. This, in turn, implies that controllers (13), (14) may be *discontinuous* in nature. To support our analysis we define state-dependent sets, $\mathcal{S}_i^o$, $\mathcal{S}_i^d$, and $\mathcal{S}_i^a$, that contain all nearby robots j such that the inter-robot distance $\|q_{ij}\|$ generates a discontinuity according to the assumptions for the objective and link maintenance potentials. Further, for convenience, we define $\widehat{\mathcal{N}}_i \triangleq \mathcal{N}_i \setminus \mathcal{S}_i^o$, $\widetilde{\mathcal{N}}_i \triangleq \mathcal{N}_i \cap \mathcal{S}_i^o$, $\widehat{\mathcal{D}}_i^d \triangleq \mathcal{D}_i^d \setminus \mathcal{S}_i^d$, $\widetilde{\mathcal{D}}_i^d \triangleq \mathcal{D}_i^d \cap$, $\mathcal{S}_i^d$, $\widehat{\mathcal{D}}_i^a \triangleq \mathcal{D}_i^a \setminus \mathcal{S}_i^a$, and $\widetilde{\mathcal{D}}_i^a \triangleq \mathcal{D}_i^a \cap \mathcal{S}_i^a$ which represent neighbor interactions without discontinuity (hat) and with discontinuity (tilde) at some $\mathbf{q} \in \mathbb{R}^{2N}$. Our first result regarding the correctness of topology control with discontinuities is now formalized.

Theorem 3.1: Consider a multi-robot system where each robot has unicycle kinematics (1) driven by control laws (13)-(14). Assume that the control functions $\psi_{ij}^o, \psi_{ij}^a, \psi_{ij}^d$ of (14) fulfill Assumption 2, yielding discontinuities in controls (13)-(14) arising at radii ρ_0, ρ_1, ρ_2 due to dynamic interactions with nearby robots. For a desired topological property $\mathbb{P}(\mathcal{G})$ and associated predicates P_{ij}^a, P_{ij}^d, if the system initial condition is such that $\mathbb{P}(\mathcal{G})$ holds, it will hold for all future time. Further, if $\psi_{ij}^o \to \infty$ as $d_{ij} \to 0$, then collisions are avoided.

Proof: Let us consider the following Lyapunov function:

$$V = \sum_{i=1}^{n}\sum_{j\neq i}\left(\psi_{ij}^{o} + \psi_{ij}^{d} + \psi_{ij}^{a}\right) \tag{16}$$

together with the set of evolution $\Omega_V = \{\mathbf{p}\,|\,V \leq c\}$ for finite $c > 0$. The level sets of V are compact and invariant with respect to the relative positions of all pairs of agents. Specifically, arguments for the compactness of Ω_V with respect to relative distance can be found for example in [6], [8], [16]. Notice that as opposed for example to the analysis of [8], the potentials fields are assumed to be only locally Lipschitz continuous and regular. This fact requires us to apply nonsmooth techniques outlined in Section II-B. First, let us define the stacked vector structure of the gradient ∇V of the Lyapunov function $\nabla V = [(\nabla V)_1^T \;\ldots\; (\nabla V)_n^T]^T$ with translational (x, y) and rotational (θ) components $(\nabla V)_i = [(\nabla V)_i^x,\, (\nabla V)_i^y,\, (\nabla V)_i^\theta]^T$, and the generalized gradient $\partial V = [(\partial V)_1^T \;\ldots\; (\partial V)_n^T]^T$ again with components $(\partial V)_i = [(\partial V)_i^x,\, (\partial V)_i^y,\, (\partial V)_i^\theta]^T$. Before computing the generalized time derivative, let us first compute the generalized gradient of (16), yielding

$$\partial V = \sum_{i=1}^{n}\Bigg(\underbrace{\sum_{j\in\widehat{\mathcal{N}}_i}\partial\psi_{ij}^{o} + \sum_{j\in\widehat{\mathcal{D}}_i^d}\partial\psi_{ij}^{d} + \sum_{j\in\widehat{\mathcal{D}}_i^a}\partial\psi_{ij}^{a}}_{\text{Non-switching}} + \underbrace{\sum_{j\in\widetilde{\mathcal{N}}_i}\partial\psi_{ij}^{o} + \sum_{j\in\widetilde{\mathcal{D}}_i^d}\partial\psi_{ij}^{d} + \sum_{j\in\widetilde{\mathcal{D}}_i^a}\partial\psi_{ij}^{a}}_{\text{Switching}}\Bigg) \tag{17}$$

under the sum rule for the generalized gradient [25], and using the fact that $\nabla_{q_i}\psi_{ij}^h = 0$, outside of $\mathcal{N}_i$, $\mathcal{D}_i^d$ and $\mathcal{D}_i^a$, respectively, with $h \in \{o,\, d,\, a\}$. The generalized gradient for each potential ψ_{ij}^h is:

$$\partial\psi_{ij}^h = \begin{cases}\overline{\text{co}}\{0_{3n},\, \nabla_{\mathbf{p}}\,\psi_{ij}^h\} & j \in \mathcal{S}_i^h\\ \nabla_{\mathbf{p}}\,\psi_{ij}^h & \text{otherwise}\end{cases} \tag{18}$$

with $\overline{\text{co}}\left\{0_{3n},\, \nabla_{\mathbf{p}}\,\psi_{ij}^h\right\} = \left\{\alpha_{ij}^h\,\nabla_{\mathbf{p}}\,\psi_{ij}^h\right\}$ where $\alpha_{ij}^h \in [0, 1]$, $0_{3n} \in \mathbb{R}^{3n}$ a vector of zeros, and the gradient $\nabla_{\mathbf{p}}\,\psi_{ij}^h$ expressed in a closed form:

$$\nabla_{\mathbf{p}}\,\psi_{ij}^h = \Big[0_3^T, \ldots, \left[(\nabla_{q_i}\psi_{ij}^h)^T, 0\right]^T, \ldots, 0_3^T, \ldots, \ldots, \left[(\nabla_{q_j}\psi_{ij}^h)^T, 0\right]^T \ldots, 0_3^T\Big]^T \tag{19}$$

Then we can describe the generalized gradient ∂V as:

$$\begin{aligned}\partial V = &\sum_{i=1}^{n}\Bigg(2\sum_{j\in\widehat{\mathcal{N}}_i}\nabla_{\mathbf{p}}\psi_{ij}^{o} + 2\sum_{j\in\widehat{\mathcal{D}}_i^d}\nabla_{\mathbf{p}}\psi_{ij}^{d} + 2\sum_{j\in\widehat{\mathcal{D}}_i^a}\nabla_{\mathbf{p}}\psi_{ij}^{a}\Bigg)\\ &+ \sum_{i=1}^{n}\Bigg(\sum_{j\in\widetilde{\mathcal{N}}_i}\Big(\overline{\text{co}}\{0_{3n},\, \nabla_{\mathbf{p}}\,\psi_{ij}^{o}\} + \overline{\text{co}}\{0_{3n},\, \nabla_{\mathbf{p}}\,\psi_{ji}^{o}\}\Big)\Bigg)\\ &+ \sum_{i=1}^{n}\Bigg(\sum_{j\in\widetilde{\mathcal{D}}_i^d}\Big(\overline{\text{co}}\{0_{3n},\, \nabla_{\mathbf{p}}\,\psi_{ij}^{d}\} + \overline{\text{co}}\{0_{3n},\, \nabla_{\mathbf{p}}\,\psi_{ji}^{d}\}\Big)\Bigg)\\ &+ \sum_{i=1}^{n}\Bigg(\sum_{j\in\widetilde{\mathcal{D}}_i^a}\Big(\overline{\text{co}}\{0_{3n},\, \nabla_{\mathbf{p}}\,\psi_{ij}^{a}\} + \overline{\text{co}}\{0_{3n},\, \nabla_{\mathbf{p}}\,\psi_{ji}^{a}\}\Big)\Bigg)\end{aligned} \tag{20}$$

where we have applied the potential symmetry properties (15). Thus, an element $\xi \in \partial V$ is of the form

$$\begin{aligned}\xi = &\sum_{i=1}^{n}\Bigg(2\sum_{j\in\widehat{\mathcal{N}}_i}\nabla_{\mathbf{p}}\psi_{ij}^{o} + 2\sum_{j\in\widehat{\mathcal{D}}_i^d}\nabla_{\mathbf{p}}\psi_{ij}^{d} + 2\sum_{j\in\widehat{\mathcal{D}}_i^a}\nabla_{\mathbf{p}}\psi_{ij}^{a}\Bigg)\\ &+ \sum_{i=1}^{n}\Bigg(\sum_{j\in\widetilde{\mathcal{N}}_i}\alpha_{ij}^{o}\,\nabla_{\mathbf{p}}\,\psi_{ij}^{o} + \alpha_{ji}^{o}\,\nabla_{\mathbf{p}}\,\psi_{ji}^{o}\Big)\Bigg)\\ &+ \sum_{i=1}^{n}\Bigg(\sum_{j\in\widetilde{\mathcal{D}}_i^d}\alpha_{ij}^{d}\,\nabla_{\mathbf{p}}\,\psi_{ij}^{d} + \alpha_{ji}^{d}\,\nabla_{\mathbf{p}}\,\psi_{ji}^{d}\Big)\Bigg)\\ &+ \sum_{i=1}^{n}\Bigg(\sum_{j\in\widetilde{\mathcal{D}}_i^a}\alpha_{ij}^{a}\,\nabla_{\mathbf{p}}\,\psi_{ij}^{a} + \alpha_{ji}^{a}\,\nabla_{\mathbf{p}}\,\psi_{ji}^{a}\Big)\Bigg)\end{aligned} \tag{21}$$

for a given selection of scalars α_{ij}^h with $h \in \{o,\, d,\, a\}$. For convenience we can also write an element $\xi \in \partial V$ as $\xi = \left[\xi_1^T,\, \ldots,\, \xi_n^T\right]^T$ where $\xi_i \in \mathbb{R}^3$ has the structure

$$\begin{aligned}\xi_i &= 2\sum_{j\in\widehat{\mathcal{N}}_i}\nabla_{p_i}\psi_{ij}^{o} + \sum_{j\in\widehat{\mathcal{N}}_i}\left(\alpha_{ij}^{o} + \alpha_{ji}^{o}\right)\nabla_{p_i}\psi_{ij}^{o}\\ &\quad + 2\sum_{j\in\widetilde{\mathcal{D}}_i^d}\nabla_{p_i}\psi_{ij}^{d} + \sum_{j\in\widehat{\mathcal{D}}_i^d}\left(\alpha_{ij}^{d} + \alpha_{ji}^{d}\right)\nabla_{p_i}\psi_{ij}^{d}\\ &\quad + 2\sum_{j\in\widetilde{\mathcal{D}}_i^a}\nabla_{p_i}\psi_{ij}^{a} + \sum_{j\in\widehat{\mathcal{D}}_i^a}\left(\alpha_{ij}^{a} + \alpha_{ji}^{a}\right)\nabla_{p_i}\psi_{ij}^{a}\\ &= \begin{bmatrix}\xi_i^x\\ \xi_i^y\\ \xi_i^\theta\end{bmatrix} \in \begin{bmatrix}(\partial V)_i^x\\ (\partial V)_i^y\\ (\partial V)_i^\theta\end{bmatrix} = \begin{bmatrix}(\partial V)_i^x\\ (\partial V)_i^y\\ \{0\}\end{bmatrix}\end{aligned} \tag{22}$$

Now we must evaluate the Filippov map of our differential equation (1). Specifically, by applying the Cartesian product rule from the calculus of Definition 2.3 we obtain:

$$K[\dot{\mathbf{p}}] \subseteq \left[K[\dot{p}_1]^T, \ldots, K[\dot{p}_n]^T\right]^T \tag{23}$$

The elements of $K[\dot{p}_i]$ then have the following form:

$$K[\dot{p}_i] \subseteq \begin{bmatrix}K[v_i\cos\theta_i]\\ K[v_i\sin\theta_i]\\ K[\omega_i]\end{bmatrix} = \begin{bmatrix}K[v_i]\cos\theta_i\\ K[v_i]\sin\theta_i\\ K[\omega_i]\end{bmatrix} \tag{24}$$

where again we have applied the Cartesian product rule, the absolute continuity of the θ_i and the continuous differentiability of the trigonometric functions. Now, according to (13), we have that $K[v_i]$ is defined as:

$$K[v_i] = -K\left[\left(f_{x_i}\cos\theta_i + f_{y_i}\sin\theta_i\right)\right] \tag{25}$$

with f_{x_i} and f_{y_i} as in (14). By exploiting the calculus we proceed as follows:

$$\begin{aligned} K[v_i] &= -K\left[\left(f_{x_i}\cos\theta_i + f_{y_i}\sin\theta_i\right)\right] \\ &\subseteq -(K\left[f_{x_i}\right]\cos\theta_i + K\left[f_{y_i}\right]\sin\theta_i) \end{aligned} \tag{26}$$

Let us now recall the chain rule given in Theorem 2.1:

$$\dot{\tilde{V}}(x(t)) = \bigcap_{\xi\in\partial V(x)} \xi^T K[f](x) \tag{27}$$

Now, for each $\xi \in \partial V$ the following holds by exploiting (23):

$$\xi^T K\left[\dot{\mathbf{p}}\right] \subseteq \sum_{i=1}^{n} \xi_i^T K\left[\dot{p}_i\right] \tag{28}$$

By performing the ith inner product of (28) we have:

$$\begin{aligned} \xi_i^T K\left[\dot{p}_i\right] &= \left[\xi_i^x \ \xi_i^y \ 0\right] \begin{bmatrix} K[v_i]\cos\theta_i \\ K[v_i]\sin\theta_i \\ K[\omega_i] \end{bmatrix} \\ &= -\left(\xi_i^x\cos\theta_i + \xi_i^y\sin\theta_i\right)\cdot \\ &\qquad \left(K\left[f_{x_i}\right]\cos\theta_i + K\left[f_{y_i}\right]\sin\theta_i\right) \end{aligned} \tag{29}$$

Let us now define the members $\hat{\xi}_i$ of the Filippov map as:

$$\hat{\xi}_i = \begin{bmatrix} \hat{\xi}_i^x \\ \hat{\xi}_i^y \\ \hat{\xi}_i^\theta \end{bmatrix} \in \begin{bmatrix} K\left[f_{x_i}\right] \\ K\left[f_{y_i}\right] \\ K\left[\omega_i\right] \end{bmatrix} = \begin{bmatrix} K[(\nabla V)_i^x] \\ K[(\nabla V)_i^y] \\ K\left[\omega_i\right] \end{bmatrix} = \begin{bmatrix} (\partial V)_i^x \\ (\partial V)_i^y \\ K\left[\omega_i\right] \end{bmatrix} \tag{30}$$

where we have exploited the calculus of Definition 2.3.

From the above it is clear that ξ_i of (22) and $\hat{\xi}_i$ of (30) are such that their projections over the x and y axes, that is $\xi_i^{xy} = [\xi_i^x \ \xi_i^y]^T$ and $\hat{\xi}_i^{xy} = [\hat{\xi}_i^x \ \hat{\xi}_i^y]^T$, are members of the same set $(\partial V)_i^{xy} = (\partial V)_i^x \times (\partial V)_i^y$, namely the x and y portion of the generalized gradient ∂V. It is this notion that allows us to reason on the negative semi-definiteness of (27). Specifically, notice that an element of the inner product given in (29) can be written as:

$$-\left(\xi_i^x\cos\theta_i + \xi_i^y\sin\theta_i\right)\left(\hat{\xi}_i^x\cos\theta_i + \hat{\xi}_i^y\sin\theta_i\right) \tag{31}$$

or equivalently in a matrix form as $-(\xi_i^{xy})^T M_i\, \hat{\xi}_i^{xy}$ with M_i defined as:

$$M_i = \begin{bmatrix} \cos^2\theta_i & \cos\theta_i\sin\theta_i \\ \cos\theta_i\sin\theta_i & \sin^2\theta_i \end{bmatrix} \tag{32}$$

with spectrum $\sigma(M_i) = \{\lambda_1 = 0, \lambda_2 = 1\}$ for all θ_i by construction. At this point, let us define ${}^\star\xi_i^{xy}$ as follows:

$${}^\star\xi_i^{xy} = \underset{\xi_i^{xy}\in(\partial V)_i^{xy}}{\arg\min} \left\{(\xi_i^{xy})^T M_i\, \xi_i^{xy}\right\} \tag{33}$$

then, the following holds for any element $\hat{\xi}_i^{xy}$:

$$({}^\star\xi_i^{xy})^T M_i\, \hat{\xi}_i^{xy} \geq 0 \tag{34}$$

It is important to notice that given the fixed eigenvalues of M_i, one may assume that $({}^\star\xi_i^{xy})^T M_i\, \hat{\xi}_i^{xy}$ is always identically zero, by choosing ${}^\star\xi_i^{xy}$ in the nullspace of M_i. However, we remind the reader that as ${}^\star\xi_i^{xy}$ is a member of a convex domain that is a *subset* of $\mathbb{R}^2$, it may not always be the null eigenvector of M_i. Note that, (34) follows from the fact that for a given convex domain D and a smooth function $f(x) = x^T Ax$ at a point $x^\star = \arg\min f(x)$, the point $x^\star$ satisfies

$$\nabla f(x^\star)(y - x^\star) \geq 0, \, \forall y \in D \tag{35}$$

Then since $\nabla f(x^\star) = 2(x^\star)^T A$, from (35) it follows that:

$$(x^\star)^T A\, y \geq (x^\star)^T A\, x^\star \geq 0, \tag{36}$$

thus proving (34).

We are now ready to reason on the Lyapunov derivative (27), which takes the form of an intersection of intervals. Seen in this way, it follows that to prove negative semi-definiteness of (27), we must identify only a single interval in the intersection $\bigcap_{\xi\in\partial V(x)} \xi^T K[f](x)$ which is negative or zero, forcing the intersection to be contained in such an interval. Given our previous analysis, we can now find such an interval as follows by recalling that $\frac{d}{dt}V(x(t)) \in^{\text{a.e.}} \dot{\tilde{V}}(x(t))$ and by noticing that by construction from (34) there always exists $\xi \in \partial V(x)$, such that $\xi_i = [{}^\star\xi_i^{xy}, 0]\ \forall\, i$ and thus for every other $\hat{\xi} \in \partial V$, where $\hat{\xi} = [\hat{\xi}_1^T, \ldots, \hat{\xi}_n^T]^T$ with $\hat{\xi}_i = [\hat{\xi}_i^{xy}, \, \hat{\xi}_i^\theta]^T$, the following holds:

$$\frac{d}{dt}V(x(t)) \leq -\sum_{i=1}^{n}({}^\star\xi_i^{xy})^T M_i\, \hat{\xi}_i^{xy} \leq 0 \tag{37}$$

Since $V(x(t))$ is absolutely continuous, $\dot{\tilde{V}}(x(t))$ is bounded below zero, and V is bounded above zero, $V(x)$ tends to a constant as $t \to \infty$. Thus, if our system is initialized in a feasible configuration with finite $V(x)$, the function remains finite for all time t. By design, our controllers approach infinity when $d_{ij} \to \rho_2^-$ for neighbors $j \in \mathcal{D}_i^d$ and $d_{ij} \to \rho_1^+$ for neighbors $j \in \mathcal{D}_i^a$. Thus, as $V(x)$ is finite for all time, decision set invariance is preserved and topological property $\mathbb{P}$ is maintained over time. Furthermore, if $\psi_{ij}^o \to \infty$ as $d_{ij} \to 0$, collision avoidance follows by equivalent reasoning. ■

IV. Saturated Actuation and Exogenous Inputs

Let us now consider the following variation of the control law given in (13) where both exogenous inputs and saturations are taken into account:

$$\begin{aligned} v_i &= -g_v\left(\left(f_{x_i} + \tilde{f}_{x_i}\right)\cos\theta_i + \left(f_{y_i} + \tilde{f}_{y_i}\right)\sin\theta_i\right) \\ \omega_i &= -g_\omega\left(\theta_i - \operatorname{arctan2}(f_{y_i} + \tilde{f}_{y_i}, f_{x_i} + \tilde{f}_{x_i})\right) \end{aligned} \tag{38}$$

where f_{x_i} and f_{y_i} are defined as (14), $\tilde{f}_{x_i}$ and $\tilde{f}_{y_i}$ are measurable and essentially locally bounded *input* functions, and $g_h(\cdot) : D \to \bar{D}$ for some convex sets $\bar{D} \subseteq D \subseteq \mathbb{R}$ is any continuous monotonic odd *saturation* function such that $|g_h(z)| \leq |z|\,, \forall\, z \in D$ with $h \in \{v, \omega\}$.

We now introduce a minor technical extension of the Filippov calculus given in Definition 2.3, which is required for computing the Filippov map under saturations as in (38).

Lemma 4.1: Consider a *continuous* and *monotonic* function $g(\cdot) : \mathbb{R} \to \mathbb{R}$ and a *measurable* and *essentially locally bounded* function $f(\cdot) : \mathbb{R} \to \mathbb{R}$, then:

$$K[g(f)](x) = \{g(K[f](x))\} \tag{39}$$

where notation $\{g(K[f](x))\}$ denotes the set of Filippov map elements $K[f](x)$ each having had $g(\cdot)$ applied to them.

Proof: Let us recall that by definition we have:

$$K[g(f)](x) = \overline{\text{co}}\{\lim g(f(x_i)) \,|\, x_i \to x,\, x_i \notin Z_f \cup Z\} \quad (40)$$

Now by exploiting a basic property of limits since $g(\cdot)$ is continuous we have that:

$$\begin{aligned}\overline{\text{co}}\{\lim g(f(x_i)) \,|\, x_i \to x,\, x_i \notin Z_f \cup Z\} = \\ \overline{\text{co}}\{g(\lim f(x_i)) \,|\, x_i \to x,\, x_i \notin Z_f \cup Z\}\end{aligned} \quad (41)$$

At this point, let $D \subset \mathbb{R}$ then if $g(\cdot)$ is monotonic we have:

$$\max_{x \in D} g(x) = g(\max_{x \in D} x) \quad \text{and} \quad \min_{x \in D} g(x) = g(\min_{x \in D} x) \quad (42)$$

which, due to the fact that we are in $\mathbb{R}$, suffices to ensure:

$$\begin{aligned}\overline{\text{co}}\{g(\lim f(x_i)) \,|\, x_i \to x,\, x_i \notin Z_f \cup Z\} = \\ \{g(\overline{\text{co}}\{\lim f(x_i) \,|\, x_i \to x,\, x_i \notin Z_f \cup Z\})\}\end{aligned} \quad (43)$$

thus proving the lemma. ■

We now prove our main result when both exogenous inputs and saturations are taken into account.

Theorem 4.1: Consider a multi-robot system where each robot has unicycle kinematics (1) driven by control laws (38) and (14). Assume that the control functions $\psi^o_{ij}, \psi^a_{ij}, \psi^d_{ij}$ of (14) fulfill Assumption 2, having discontinuities arising at radii ρ_0, ρ_1, ρ_2 due to dynamic interactions with nearby robots. For a desired topological property $\mathbb{P}(\mathcal{G})$ and associated predicates P^a_{ij}, P^d_{ij}, if the system initial condition is such that $\mathbb{P}(\mathcal{G})$ holds, it will hold for all future time. Further, if $\psi^o_{ij} \to \infty$ as $d_{ij} \to 0$, then collisions are avoided.

Proof: The proof follows similar reasoning as in Theorem 3.1. In particular, we consider again the Lyapuov function given in (16) for which, according to (38) and by exploiting Lemma 4.1, $K[v_i]$ now takes the following form:

$$\begin{aligned}K[v_i] &= -K\left[g\Big((f_{x_i} + \tilde{f}_{x_i})\cos\theta_i + (f_{y_i} + \tilde{f}_{y_i})\sin\theta_i\Big)\right] \\ &= -g\left(\Big(K[f_{x_i}] + K[\tilde{f}_{x_i}]\Big)\cos\theta_i \right.\\ &\qquad\left. + \Big(K[f_{y_i}] + K[\tilde{f}_{y_i}]\Big)\sin\theta_i\right)\end{aligned} \quad (44)$$

The analysis now involves determining the structure of the intersection (27) as follows:

$$\begin{aligned}\xi_i^T \dot{p}_i &= [\xi^x_i \; \xi^y_i \; 0]\begin{bmatrix}K[v_i]\cos\theta_i \\ K[v_i]\sin\theta_i \\ K[\omega_i]\end{bmatrix} \\ &= -\big(\xi^x_i\cos\theta_i + \xi^y_i\sin\theta_i\big)\cdot \\ &\quad g\left(\Big(K[f_{x_i}] + K[\tilde{f}_{x_i}]\Big)\cos\theta_i \right.\\ &\qquad\left. + \Big(K[f_{y_i}] + K[\tilde{f}_{y_i}]\Big)\sin\theta_i\right)\end{aligned} \quad (45)$$

where $\{f_{x_i},\, f_{y_i}\}$ and $\{\tilde{f}_{x_i},\, \tilde{f}_{y_i}\}$ are respectively nominal and exogenous contributions. By recalling the chain rule of Theorem 2.1 having form (28) we obtain:

$$\begin{aligned}\xi^T K[\dot{\mathbf{p}}] \subseteq \sum_{i=1}^n \xi_i^T K[\dot{p}_i] \subseteq -\sum_{i=1}^n \Big[\big(\xi^x_i\cos\theta_i + \xi^y_i\sin\theta_i\big)\cdot \\ g\left(\Big(K[f_{x_i}] + K[\tilde{f}_{x_i}]\Big)\cos\theta_i + \Big(K[f_{y_i}] + K[\tilde{f}_{y_i}]\Big)\sin\theta_i\right)\Big]\end{aligned} \quad (46)$$

An element of the above intervals will then have the form:

$$\begin{aligned}-\sum_{i=1}^n \big(\xi^x_i\cos\theta_i + \xi^y_i\sin\theta_i\big)\cdot \\ g\Bigg(\hat{\xi}^x_i\cos\theta_i + \hat{\xi}^y_i\sin\theta_i + \underbrace{\tilde{\xi}^x_i\cos\theta_i + \tilde{\xi}^y_i\sin\theta_i}_{z}\Bigg)\end{aligned} \quad (47)$$

where $\tilde{\xi}^x_i \in K[\tilde{f}_{x_i}]$ and $\tilde{\xi}^y_i \in K[\tilde{f}_{y_i}]$.

Clearly the form of (47) is closely related to the result derived in Theorem 3.1. It is therefore reasonable to proceed by invoking reasoning from Theorem 3.1 in order to prove our result in this context. In that direction, first consider an equivalent form for the inner product element (31), where instead of a quadratic form we work simply with the scalar multiplication $-\gamma_i\hat{\gamma}_i$ where $\gamma_i, \hat{\gamma}_i \in \mathbb{R}$. Notice that $\xi^{xy}_i = [\xi^x_i, \xi^y_i]^T, \hat{\xi}^{xy}_i = [\hat{\xi}^x_i, \hat{\xi}^y_i]^T$ are members of the same convex domain $D \subseteq \mathbb{R}$ and for a fixed θ_i, the terms $\gamma_i \triangleq \xi^x_i\cos\theta_i + \xi^y_i\sin\theta_i$ and $\hat{\gamma}_i \triangleq \hat{\xi}^x_i\cos\theta_i + \hat{\xi}^y_i\sin\theta_i$ are linear combinations and thus themselves members of the same convex domain $D' \subseteq \mathbb{R}$. The reasoning applied in Theorem 3.1 is then restated as:

$$\exists\, \gamma^\star_i \in D' \mid \gamma^\star_i\hat{\gamma}_i \geq \gamma^\star_i\gamma^\star_i,\; \forall\, \hat{\gamma}_i \in D' \quad (48)$$

Now, consider the following logical implication which we would like to prove true:

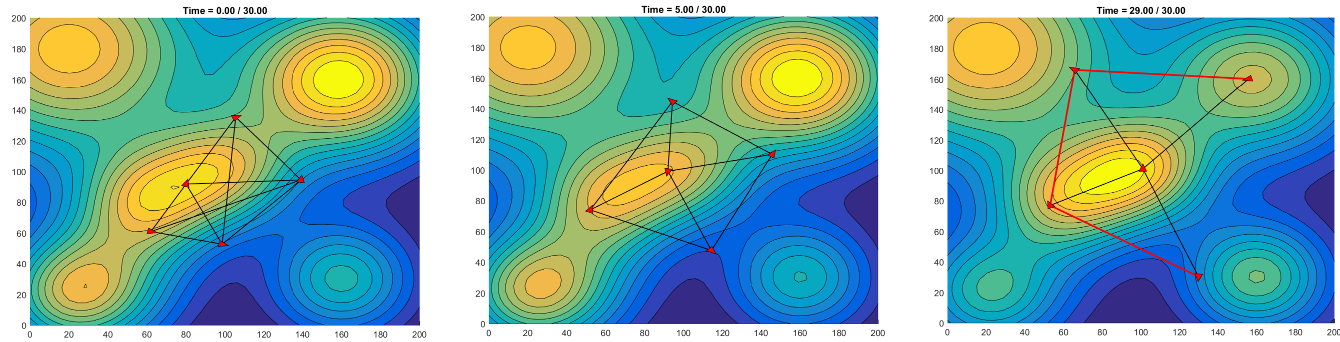
$$\begin{aligned}\exists\, \gamma^\star_i \in D' \mid \gamma^\star_i\hat{\gamma}_i \geq \gamma^\star_i\gamma^\star_i,\; \forall\, \hat{\gamma}_i \in D' \to \\ \gamma^\star_i g(\hat{\gamma}_i + z) \geq \gamma^\star_i g(\gamma^\star_i + z),\; \forall\, \hat{\gamma}_i \in D', z \leq \mathcal{Z}\end{aligned} \quad (49)$$

The above statement, which will lead us to our result, is proven by contradiction as follows. Assume that the statement $\exists\, \gamma^\star_i \in D' \mid \gamma^\star_i\hat{\gamma}_i \geq \gamma^\star_i\gamma^\star_i,\; \forall\, \hat{\gamma}_i \in D'$ holds, and that there exists some $\hat{\gamma}_i \in D'$ such that $\gamma^\star_i g(\hat{\gamma}_i + z) < \gamma^\star_i g(\gamma^\star_i + z)$ for some $z \leq \mathcal{Z}$. If $\gamma^\star_i > 0$, we have that $\gamma^\star_i g(\hat{\gamma}_i + z) < \gamma^\star_i g(\gamma^\star_i + z)$ implies $\hat{\gamma}_i + z < \gamma^\star_i + z \to \hat{\gamma}_i < \gamma^\star_i$ due to the non-decreasingness of $g(\cdot)$, which is clearly contradictory to $\gamma^\star_i\hat{\gamma}_i \geq \gamma^\star_i\gamma^\star_i,\; \forall\, \hat{\gamma}_i \in D'$. If instead we have $\gamma^\star_i < 0$, an analogous contradiction arises. Thus, our assumption that there exists some $\hat{\gamma}_i \in D'$ such that $\gamma^\star_i g(\hat{\gamma}_i + z) < \gamma^\star_i g(\gamma^\star_i + z)$ is incorrect, and thus the statement (49) is verified. To complete the proof, we simply must ensure that the sign of (47) eventually becomes negative as the system approaches undesired configurations. This is equivalent to requiring that $\gamma^\star_i$ and $g(\gamma^\star_i + z)$ in the above general reasoning have the same sign and that their product becomes arbitrarily large (i.e., we view them as some element of (47)). To prove this last part, let us recall that for any $\xi \notin \text{null}(M_i)$, since $\sigma(M_i) = \{0, 1\}$, it follows that ξ must be aligned with the eigenvector associated to $\lambda_2 = 1$, and thus:

$$-\sum_{i=1}^n {}^\star\xi^{xy\,T}_i M_i\, {}^\star\xi^{xy}_i = -\sum_{i=1}^n \|{}^\star\xi^{xy}_i\|^2 = -\|{}^\star\xi^{xy}\|^2 \quad (50)$$

where the eigenvector decomposition was used to obtain the first equality.

Then, if $\|{}^\star\xi^{xy}\|$ goes to infinity, then there always exists a $\|{}^\star\xi^{xy}_i\|$ that goes to infinity, and thus for the equivalence seen before, also $\big({}^\star\xi^x_i\cos\theta_i + {}^\star\xi^y_i\sin\theta_i\big)$ goes to infinity. At



(a) (b) (c)

Fig. 1. Control of graph rigidity with actuator saturation and discontinuously arriving environmental sensing objective.

this point, by recalling that $\gamma_i^\star \triangleq \left({}^\star\xi_i^x \cos\theta_i + {}^\star\xi_i^y \sin\theta_i\right)$ (for some i), this implies that being $z \leq \mathcal{Z}$, $\gamma_i^\star$ will eventually dominate $\mathcal{Z}$ in magnitude as it becomes arbitrarily large, and thus the terms $\gamma_i^\star$ and $g(\gamma_i^\star + z)$ will have the same sign. Also, it is clear that for the same reason, $\gamma_i^\star\, g(\gamma_i^\star + z)$ will grow arbitrarily large, and thus dominating in magnitude the entire sum in (47). Finally, recalling that the intervals (46) lie in an intersection due to the chain rule Theorem 2.1, we conclude that as the system approaches undesired configurations $\dot{\tilde{V}} \leq 0$. As in Theorem 3.1, this fact along with our control function construction yields our result. ■

V. Simulation Results

As our analysis has shown the correctness of topology control for arbitrary exogenous objectives with discontinuities, we choose here the objective of environmental sensing for simulations (e.g., as in [26]). In particular, we model incoming adaptive sensing commands as *discontinuous exogenous objectives* to mimic what may occur when a centralized entity computes and relays commands to a decentralized team. Furthermore, we assume robots to be capable of measuring the environmental process, e.g., in the case of a thermal plume or air pollution dispersion the robots may measure temperature or a relevant chemical signature, respectively.

Consider a workspace in $\mathbb{R}^2$ within which a *time-varying* environmental process $\varphi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ evolves. It is common to model a time-varying process as a linear combination of fixed basis functions with time-varying weights, [27]. That is $\varphi(x,t) = \sum_{i=1}^{B} r_i(t) b_i(x) = [\mathbf{r}(t)]^T \mathbf{b}(x)$ where $r_i : \mathbb{R} \to \mathbb{R}$ are the time-varying weights for the basis functions $b_i : \mathbb{R}^2 \to \mathbb{R}$, for all $i = 1, \ldots, B$, with $\mathbf{r} \triangleq [q_1, \ldots, q_B]^T \in \mathbb{R}^B$ and $\mathbf{b} \triangleq [b_1, \ldots, b_B]^T \in \mathbb{R}^B$ the stacked vectors of weights and basis functions, respectively. The evolution of the weights is then described by the noisy linear dynamics $\dot{\mathbf{r}} = A\mathbf{r} + \mathbf{w}(t)$ for some fixed transition matrix $A \in \mathbb{R}^{B\times B}$ and zero-mean Gaussian noise $\mathbf{w}(t) : \mathbb{R} \to \mathbb{R}^B$. Our sensing objective is to choose team inputs that allow for the best estimation of the weight process $\mathbf{r}(t)$, which given a known set of basis functions, yields an estimate of the environmental process φ. Given that our weight dynamics are linear, the optimal estimator is the *Kalman filter*, the details of which are well-known and thus omitted here. Denoting by $H \in \mathbb{R}^{B\times B}$ the covariance matrix of the Kalman filter applied to the weights of the environmental model, our exogenous environmental sensing objective takes the final form $\tilde{f} = \arg\min_{\mathbf{u}} \left[\mathrm{Trace}(H^+(\mathbf{u}))\right]$ where $H^+(\mathbf{u})$ is the covariance matrix for a simulated step forward of the Kalman filter under unicycle kinematics (1) with stacked velocity command $\mathbf{u} = [u_1, \ldots, u_n]^T$ and weight dynamics introduced above. Such a process may yield discontinuously arriving exogenous commands due to a stepwise approximated optimization or the fact that the optimization may run orders of magnitude slower than topology control.

We now only require a description of the topological constraints for the multi-robot team, i.e., *rigidity* of the sensing graph. Briefly, rigidity represents an important requirement of a multi-robot interaction graph, as it is necessary for information flow and localizability [28]. Following the requirements outlined in Section III-A, the deletion predicates for rigidity control are defined as $P_{ij}^d \triangleq f_{\text{rigid}}(\mathcal{G}_w)$ where f_{rigid} is a Boolean function indicating the rigidity (computed by applying decentralized algorithm [10]) of the worst-case graph $\mathcal{G}_w$ given by $\mathcal{G}_w = (\mathcal{V}, \mathcal{E}_w)$, $\mathcal{E}_w \triangleq \{(i,j) \in \mathcal{E} \,|\, i \in \mathcal{V}, j \notin \mathcal{C}_i^d\}$ with $\mathcal{C}_i^d$ the set of neighbors of agent i for which a decision to delete the edge (i,j) has been made. For the proposed objective, link addition is not a concern. However, we reiterate that our formulation is completely general and can accommodate many interesting topological properties beyond rigidity.

Our simulation results for rigidity control with an environmental sensing objective are reported in Fig.1a-1c. Each unicycle robot is represented by an isosceles triangle, while the edges of the interaction graph are denoted by black lines for uncontrolled edges and red lines for controlled edges. The level curves of the time-varying process are shown with color indicating the process value, where we have used five Gaussian basis functions ($B = 5$) distributed in the environment (note the Gaussian means depicted in Fig.1a). The remaining parameter settings for the simulation were: Gaussian variances $\{25, 25, 30, 35, 25\}$, *fixed* weights $\{5, 5, 3, 8, 4\}$ for the Gaussian bases, and time-varying weight matrix $A = -0.05\mathbf{I}_5$ with initial condition $\mathbf{r} = \mathbf{1}$. The results of Fig.1 indeed verify our theoretical contributions. Rigidity is maintained over the team trajectories, despite the presence of a discontinuously arriving exogenous objective and saturated actuations. Furthermore, we continue to see an expected behavior from our sensing objective. As the most variation in the weight process, and thus the most information,

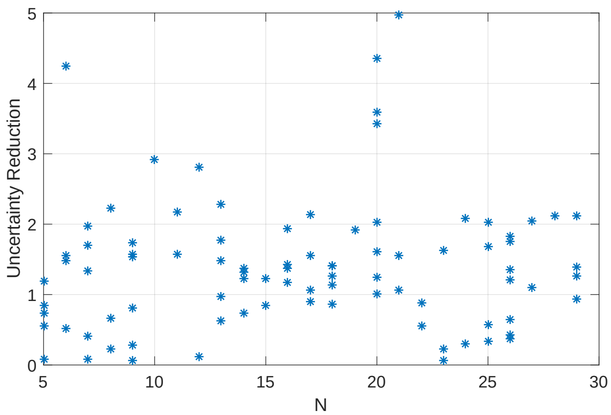


Fig. 2. Monte Carlo analysis comparing network size n to uncertainty reduction (measured as trace(H^+) normalized by the number of basis functions B), from the initial to final team configuration.

is found at the basis function centers, the team seeks the centers while guaranteeing the topological (rigidity) constraint.

Finally, we report in Fig.2 the results of a Monte Carlo study of the above example. Specifically, we run 100 randomized instances of the environmental sensing objective with rigidity maintenance. For each instance, random network size n, initial agent positions, environmental process weights $\mathbf{r}$, Gaussian basis functions $\mathbf{b}$, and transition matrix A were selected. Additionally, we guaranteed that the initial graph topology was rigid. Now, Fig.2 depicts the reduction in uncertainty of the Kalman filter (as measured by trace(H^+) normalized by the number of basis functions B) between the initial configuration of the system and the final one, for all 100 random instances. Rigidity was verified as being maintained for each instance, while Fig.2 demonstrates that estimator uncertainty was substantially reduced. In other words, this suggests that although in general topology control constrains team motion, interesting exogenous objectives can still be faithfully executed.

VI. Conclusions

In this paper, we considered the problem of *general* topology control in multi-robot systems with nonholonomic kinematics. We provided the extension of a mobility-based link control framework to nonholonomic kinematics, by demonstrating the correctness of topology control when interactions switch *arbitrarily* and when potential-based mobility is *discontinuous* with respect to topology changes. Further, we demonstrated that a multi-robot team under the above listed conditions continues to achieve topology control when actuator saturation is applied and in the presence of arbitrary discontinuous (and possibly non-pairwise) exogenous objectives.

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